

Algorithms for the Undecided – Homework 6

Discussed on Friday, 8th July, 2016.

Exercise 6.1 **Sturm Sequence**

Build the Sturm sequence of the following polynomial:

$$P = X^4 - 4X^3 - X^2 + 16X - 12$$

Count the number of roots of P in \mathbb{R} with the help of the Sturm sequence.

Hint: All polynomials in the Sturm sequence can be multiplied with positive constants without changing the outcome of the algorithm and for getting nicer looking integer coefficients in the polynomials.

Solution: The sturm sequence:

$$\begin{aligned} P_0 &= P = X^4 - 4X^3 - X^2 + 16X - 12 \\ P_1 &= P' = 4X^3 - 12X^2 - 2X - 16 \\ P_2 &= -(P_0 \bmod P_1) = \frac{7}{2}X^2 - \frac{23}{2}X + 8 = \frac{1}{2}(7X^2 - 23X + 16) \\ P_3 &= \frac{362}{49}X + \frac{565}{49} = \frac{1}{49}(362X + 565) \\ P_4 &= \frac{44100}{32761} \end{aligned}$$

The sign changes at $-\infty$ and $+\infty$:

	$-\infty$	$+\infty$
P_0	+	+
P_1	-	+
P_2	+	+
P_3	-	+
P_4	+	+

There are four sign changes for $-\infty$ and zero sign changes for $+\infty$. So P has $4 - 0 = 4$ roots. ($P(x) = 0$ for $x = -2$ or $x = 1$ or $x = 2$ or $x = 3$)

Exercise 6.2 **Sturm Sequence**

Consider the following equation and inequality:

$$\begin{aligned} x^3 + 3x^2 - 4x - 12 &= 0 \\ -x &> 0 \end{aligned}$$

Use the method from the seminar with Sturm sequences to check if there exists a real solution to this system.

Hint: You only need to show the existence of a solution.

Solution: Building the Sturm sequence for $P = x^3 + 3x^2 - 4x - 12$ and $Q = -x$:

$$\begin{aligned}
 P_0 &= P = x^3 + 3x^2 - 4x - 12 \\
 P_1 &= P'Q = (3x^2 + 6x - 4)(-x) \\
 P_2 &= -(P_0 \bmod P_1) = \frac{1}{3}(-3x^2 + 8x + 36) \\
 P_3 &= -(P_1 \bmod P_2) = \frac{1}{3}(208x + 504) \\
 P_4 &= -(P_2 \bmod P_3) = \frac{225}{676}
 \end{aligned}$$

	-∞	∞
P_0	-	+
P_1	+	-
P_2	-	-
P_3	-	+
P_4	+	+

Let $v_{P,Q}$ be the sign changes at $-\infty$ minus the sign changes at ∞ . Let $c_{>0}$ be the number of roots of P where $Q(x) > 0$ and $c_{<0}$ be the number of roots of P where $Q(x) < 0$, then $v_{P,Q} = c_{>0} - c_{<0}$

Then build the Sturm sequence for $P = x^3 + 3x^2 - 4x - 12$ and $Q^2 = (-x)^2$:

$$\begin{aligned}
 P_0 &= P = x^3 + 3x^2 - 4x - 12 \\
 P_1 &= P'Q^2 = (3x^2 + 6x - 4)(-x)^2 \\
 P_2 &= -(P_0 \bmod P_1) = x^3 - 3x^2 + 4x + 12 \\
 P_3 &= -17x^2 - 24x + 36 \\
 P_4 &= \frac{1}{289}(-1192x - 2496) \\
 P_5 &= -\frac{260100}{22201}
 \end{aligned}$$

	-∞	∞
P_0	-	+
P_1	+	+
P_2	-	+
P_3	-	-
P_4	+	-
P_5	-	-

Let v_{P,Q^2} be the sign changes at $-\infty$ minus the sign changes at ∞ of this Sturm sequence. v_{P,Q^2} is the number of roots of P where $Q^2(x) > 0$ (since $Q^2(x)$ can't be smaller than zero), so $v_{P,Q^2} = c_{>0} + c_{<0}$.

So $c_{>0} = \frac{1}{2}(v_{P,Q^2} + v_{P,Q}) = \frac{1}{2}((4 - 1) + (3 - 2)) = 2$.
 \implies There are two x which satisfy $P(x) = 0$ and $Q(x) > 0$.

Exercise 6.3 System of inequalities

Please ignore this exercise.

Use the method described on the slides to decide if there exists a real solution to the following system of inequalities:-

$$\begin{aligned}
 & \frac{x^2}{x^2+1} + \frac{x}{x^2+1} > 0 \\
 & \frac{x}{x^2+1} > 0
 \end{aligned}$$

Exercise 6.4 Quantifier elimination

Use quantifier elimination to transform the following formula to a system depending only on a, b and c :

$$\exists x. ax^2 + bx + c = 0$$

Solution: We follow the algorithm from the handout.

Case 1: $a \neq 0$

$$P_0 = ax^2 + bx + c$$

$$P_1 = 2ax + b$$

$$P_2 = \frac{b^2}{4a} - c$$

	$a > 0$		$a > 0$		$a < 0$		$a < 0$
	$P_4 > 0$		$P_2 < 0$		$P_2 > 0$		$P_2 < 0$
	$-\infty \quad \infty$		$-\infty \quad \infty$		$-\infty \quad \infty$		$-\infty \quad \infty$
P_0	+ +	P_0	+ +	P_0	- -	P_0	- -
P_1	- +	P_1	- +	P_1	+ -	P_1	+ -
P_2	+ +	P_2	- -	P_2	+ +	P_2	- -
	2 roots		0 roots		0 roots		2 roots

So there are two roots if $(a > 0 \wedge \frac{b^2}{4a} - c > 0) \vee (a < 0 \wedge \frac{b^2}{4a} - c < 0)$. If $a < 0$ then $\frac{b^2}{4a} - c < 0 \iff b^2 - 4ac > 0$ and we can simplify the formula to $a \neq 0 \wedge b^2 - 4ac > 0$.

Case 2: $a = 0$

Case 2.1: $b \neq 0$

Does $bx + c = 0$ have a solution in x?

$$P_0 = bx + c$$

$$P_1 = b$$

	$b > 0$		$b < 0$
	$-\infty \quad \infty$		$-\infty \quad \infty$
P_0	- +	P_0	+ -
P_1	+ +	P_1	- -
	1 root		1 root

So there is a solution if $a = 0 \wedge b \neq 0$.

Case 2.2: $b = 0$

$c = 0$ is obviously only true if $c = 0$.

Conclusion:

$$\begin{aligned} \exists x. ax^2 + bx + c = 0 \\ \iff \\ (a \neq 0 \wedge b^2 - 4ac > 0) \vee \\ (a = 0 \wedge b \neq 0) \vee \\ (a = 0 \wedge b = 0 \wedge c = 0) \end{aligned}$$

Feel free to ask questions at haslbecm@in.tum.de.