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Lambda Calculus
Winter Term 2017/18
Exercise Sheet 6

Exercise 1 (Reduction Relation with Closures)

For the evaluation of lambda terms that is closer to evaluation of programs in functional programming languages, one usually replaces textual substitution $t[v/x]$ with a more lazy approach that records the binding $x \mapsto v$ in an environment. These bindings are used whenever need the value of a variable v.

In this approach abstractions $\lambda x.t$ do not evaluate to themselves, but to a pair $(\lambda x.t)[e]$, where e is the current environment. We call such pairs function *closures*.

a) Define a big-step reduction relation for the lambda calculus with function closures and environments.

b) Add explicit error handling for the case where the biding of a free variable v cannot be found in the enviroment. Introduce an explicit value **abort** to indicate such an exception in the relation.

Exercise 2 (Reduction Relation with Pattern Matching)

In this exercise, we consider a λ-calculus extended with a special set of constructor values and pattern matching. Constructor values are constructed according to the following grammar:

$$c \ ::= \ C\ (c_1, \ldots, c_n) \text{ for } n \geq 0$$

where C is one from a distinguished set of *constructor* symbols.

We illustrate pattern matching by example. The expression

$$\textbf{match } C_1\ (\mathsf{false}) \textbf{ with } C_2\ () \to \mathsf{true} \mid C_1\ (x) \to x$$

should evaluate to false, while

$$\textbf{match } C_2\ (\mathsf{false}) \textbf{ with } C_2\ () \to \mathsf{true} \mid C_1\ (x) \to x$$

should evaluate to **abort**.

a) Define a big-step reduction relation for this language.

b) Prove that the two derivations stated informally above are indeed possible in the relation.

Homework 3 (Normal Forms)

Recall the inductive definition of the set NF of *normal forms*:

$$\frac{t \in \mathrm{NF}}{\lambda x.t \in \mathrm{NF}}$$

$$\frac{n \geq 0 \qquad t_1 \in \mathrm{NF} \qquad t_2 \in \mathrm{NF} \qquad \dots \qquad t_n \in \mathrm{NF}}{x\ t_1\ t_2\ \dots\ t_n \in \mathrm{NF}}$$

Show that this set precisely captures all normal forms, i.e.:

$$t \in \mathrm{NF} \Leftrightarrow \nexists t'.t \rightarrow_\beta t'$$

Homework 4 (Normal Forms & Big Step)

Show:

$$t \in \mathrm{NF} \wedge t \Rightarrow_n u \Longrightarrow u = t$$

Homework 5 (Proofs with Small-steps and Big-steps)

Let $\omega := \lambda x.xx$ and

$$t \ := \ (\lambda x.(\lambda xy.x)zy)(\omega\omega((\lambda xy.x)y))\,.$$

Prove the following:

a) $t \Rightarrow_n z$

b) $t \rightarrow^3_{cbv} t$

c) $t \not\rightarrow^+_{cbn} t$