

Exercise 1 (Better Translation Algorithm)

Give a variant of the translation algorithm that produces shorter terms. More specifically, define a variant of $\lambda^*x.t$ that analyzes more precisely where x actually appears in t .

Solution

$$\begin{array}{ll}
 \lambda^*x.x = & I \\
 \lambda^*x.X = & \mathbf{K}X \text{ if } x \notin FV(X) \\
 \lambda^*x.Xx = & X \text{ if } x \notin FV(X) \\
 \lambda^*x.(XY) = & \mathbf{B}X(\lambda^*x.Y) \text{ if } x \notin FV(X) \wedge x \in FV(Y) \\
 \lambda^*x.(YX) = & \mathbf{C}(\lambda^*x.Y)X \text{ if } x \notin FV(X) \wedge x \in FV(Y) \\
 \lambda^*x.(XY) = & \mathbf{S}(\lambda^*x.X)(\lambda^*x.Y) \text{ if } x \in FV(X, Y)
 \end{array}$$

where $\mathbf{B} := \mathbf{S}(\mathbf{K}\mathbf{S})\mathbf{K}$ and $\mathbf{C} := \mathbf{S}(\mathbf{B}\mathbf{B}\mathbf{S})(\mathbf{K}\mathbf{K})$. \mathbf{B} and \mathbf{C} fulfill the following properties

$$\begin{array}{l}
 \mathbf{B}XYZ \rightarrow^* X(YZ) \\
 \mathbf{C}XYZ \rightarrow^* XZY
 \end{array}$$

Exercise 2 (Confluence of \rightarrow)

Introduce a parallel reduction relation $>$ for combinatory logic and show that it fulfills the diamond property.

Solution

We copy the rules for \mathbf{I} , \mathbf{K} , and \mathbf{S} from \rightarrow and add:

$$\begin{array}{c}
 X > X \\
 \frac{X > X' \quad Y > Y'}{XY > X'Y'}
 \end{array}$$

We prove $X_s > Y_1 \wedge X_s > Y_2 \implies \exists X_u. Y_1 > X_u \wedge Y_2 > X_u$ by induction on the first derivation of $X_s > Y_1$. The cases for \mathbf{I} , \mathbf{K} , and \mathbf{S} are all similar, so we only consider the latter. We have $X_s = \mathbf{S} X Y Z > (X Z) (Y Z) = Y_1$. We analyze the derivation of $X_s > Y_2$. The cases for \mathbf{I} and \mathbf{K} do not apply, and the case for \mathbf{S} is trivial (choose $X_u = Y_1$). In the case of the application rule, we get $X > X'$, $Y > Y'$, $Z > Z'$ for some X' , Y' , Z' .

Choose $X_u = (X' Z') (Y' Z')$. Then $Y_1 > X_u$ by the application rule and $Y_2 > X_u$ by the S-rule.

For the application rule, we have $X_s = X Y > X' Y' = Y_1$, $X > X'$, and $Y > Y'$. For $X_s > Y_2$ all derivations are possible. We have considered all the cases before, only the case of the application rule remains. Thus we have $Y_2 > X'' Y''$, $X > X''$, and $Y > Y''$. With the induction hypothesis we obtain Y_x and Y_y such that $X' > Y_x \wedge X'' > Y_x$ and $Y' > Y_y \wedge Y'' > Y_y$. We choose $X_u = Y_x Y_y$ and conclude $Y_1 > X_u$ and $Y_2 > X_u$ by using the application rule.

Homework 3 (More Combinators)

Find combinators O and W such that:

$$\begin{aligned} O &\rightarrow^+ O \\ WXY &\rightarrow^* XYY \end{aligned}$$

Homework 4 (Mocking Birds)

Consider a combinatory logic that only provides the basic combinators B and M (the “mocking bird”) where:

$$\begin{aligned} BXYZ &\rightarrow^* X(YZ) \\ MX &\rightarrow^* XX \end{aligned}$$

Prove the following properties of this logic:

- a) For every combinator X , there is a combinator Y such that $Y \rightarrow^* XY$.
- b) For all combinators U and W , there exist combinators X and Y such that $Y \rightarrow^* UX$ and $X \rightarrow^* WY$.

Homework 5 (A Different Basis)

Show that B , C , K and W are also sufficient to represent all λ -terms, where

$$CXYZ \rightarrow XZY$$

Homework 6 (Correctness of the Translation Algorithm)

Show that the translation algorithm given in the tutorial is correct. That is, show that it fulfills the following property:

$$(\lambda^*x. X)Y \rightarrow^* X[Y/x]$$