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Lambda Calculus
Winter Term 2021/22
Exercise Sheet 4

Exercise 1 (Confluence of β-Reduction)

In the lecture we have shown the confluence of $\longrightarrow_\beta$ using the diamond property of parallel β-reduction. In this exercise, we develop an alternative proof.

We define the operation $*$ on λ-terms inductively over the structure of terms:

$$\begin{aligned} x^* &= x \\ (\lambda x.\ t)^* &= \lambda x.\ t^* \\ (t_1\ t_2)^* &= t_1^*\ t_2^* \qquad \text{if } t_1\ t_2 \text{ is not a } \beta\text{-redex.} \\ ((\lambda x.\ t_1)\ t_2)^* &= t_1^*[t_2^*/x] \end{aligned}$$

a) Show that we have for two arbitrary λ-terms s and t: $s > t \implies t > s^*$

b) Show that $\longrightarrow_\beta$ is confluent.

Exercise 2 (Parallel Beta Reduction)

Show:

$$s > t \Longrightarrow s \longrightarrow_\beta^* t$$

Exercise 3 (Predecessor and Tail)

a) Define a predecessor function pred on church numerals.

b) Use the same idea to define tl on the list encoding from homework 2.5.

Homework 4 (Parallel Beta Reduction & Substitution)

Show:

$$s > s' \wedge t > t' \Longrightarrow s[t/x] > s'[t'/x]$$

Homework 5 (Equivalence modulo β-conversion)

Assume that we add the additional axiom

$$\lambda x\ y.\ x =_\beta \lambda x\ y.\ y$$

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a) Show that under this assumption $t =_\beta t'$ for all t, t'.

b) Repeat the same for the axiom $\lambda x.\ x =_\beta \lambda x\ y.\ y\ x$.

Homework 6 (Böhm's Theorem)

Böhm's Theorem states that for arbitrary closed terms $M \neq N$ without constant atoms in $\beta\eta$-normal form, there exist $n \geq 0$ and $L_1, \ldots, L_n$ such that:

$$M\ L_1\ \ldots\ L_n\ x\ y \to_\beta^* x \text{ and } N\ L_1\ \ldots\ L_n\ x\ y \to_\beta^* y\,.$$

That is, we can tell M and N apart. Show the following two special cases:

a) $M = \lambda x\ y\ z.\ x\ z\ (y\ z)$ and $N = \lambda x\ y\ z.\ x\ (y\ z)$

b) $M = \lambda x\ y.\ x\ (y\ y)$ and $N = \lambda x\ y.\ x\ (y\ x)$