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Lambda Calculus
Winter Term 2022/23
Exercise Sheet 7

Exercise 1 (Reduction Relation with Closures)

For the evaluation of lambda terms that is closer to evaluation of programs in functional programming languages, one usually replaces textual substitution $t[v/x]$ with a more lazy approach that records the binding $x \mapsto v$ in an environment. These bindings are used whenever we need the value of a variable v.

In this approach abstractions $\lambda x.\ t$ do not evaluate to themselves, but to a pair $(\lambda x.\ t)[e]$, where e is the current environment. We call such pairs function *closures*.

a) Define a big-step reduction relation for the lambda calculus with function closures and environments.

b) Add explicit error handling for the case where the binding of a free variable v cannot be found in the enviroment. Introduce an explicit value **abort** to indicate such an exception in the relation.

Exercise 2 (Better Translation Algorithm)

Give a variant of the translation algorithm that produces shorter terms. More specifically, define a variant of $\lambda^* x.\ t$ that analyzes more precisely where x actually appears in t.

Homework 3 (Proofs with Small-steps and Big-steps)

Let $\omega := \lambda x.\ x\ x$ and

$$t\ :=\ (\lambda x.\ (\lambda x\ y.\ x)\ z\ y)\ (\omega\ \omega\ ((\lambda x\ y.\ x)\ y))\,.$$

Prove the following:

a) $t \Rightarrow_n z$

b) $t \rightarrow^3_{cbv} t$

c) $t \not\rightarrow^+_{cbn} t$

Homework 4 (More Combinators)

Find combinators O and W such that:

$$\mathsf{O}\ \rightarrow^+ \mathsf{O}$$
$$\mathsf{W}\,X\,Y \rightarrow^* X\,Y\,Y$$

Homework 5 (Mocking Birds)

Consider a combinatory logic that only provides the basic combinators B and M (the "mocking bird") where:

$$\mathsf{B}\ X\ Y\ Z \rightarrow^* X\ (Y\ Z)$$
$$\mathsf{M}\ X \rightarrow^* X\ X$$

Prove the following properties of this logic:

a) For every combinator X, there is a combinator Y such that $Y \rightarrow^* X\ Y$.

b) For all combinators U and W, there exist combinators X and Y such that $Y \rightarrow^* U\ X$ and $X \rightarrow^* W\ Y$.

Homework 6 (Correctness of the Translation Algorithm)

Show that the translation algorithm given in the tutorial is correct. That is, show that it fulfills the following property:

$$(\lambda^* x.\ X)\,Y \rightarrow^* X[Y/x]$$