

Exercise 1 (Confluence & Commutation)

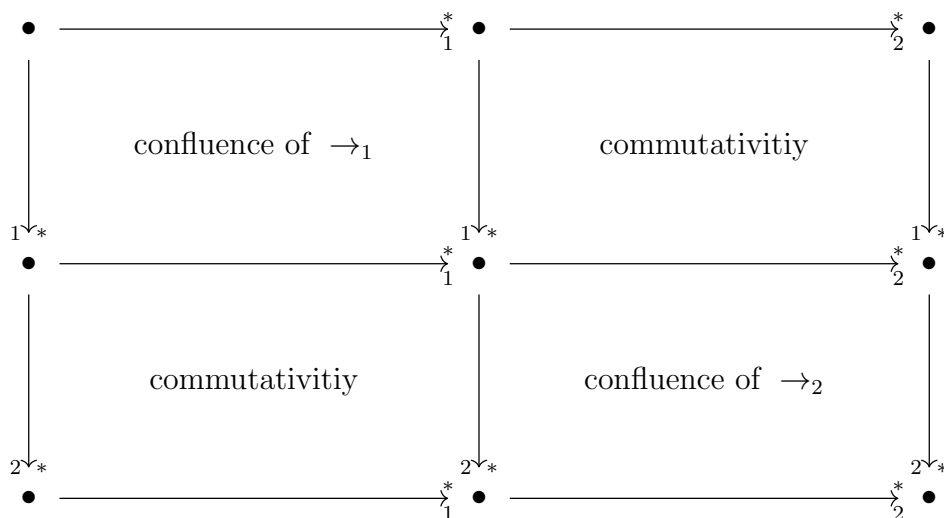
Show: If \rightarrow_1 and \rightarrow_2 are confluent, and if \rightarrow_1^* and \rightarrow_2^* commute, then $\rightarrow_{12} := \rightarrow_1 \cup \rightarrow_2$ is also confluent.

Solution

Lemma A.3.2 from the lecture. The key idea is to consider $\rightarrow_1^* \circ \rightarrow_2^*$ as \rightarrow_{12}^* unfolds into iterations of this relation, i.e. $(\rightarrow_1^* \circ \rightarrow_2^*)^* = \rightarrow_{12}^*$. More precisely:

$$\rightarrow_{12} \subseteq \rightarrow_1^* \circ \rightarrow_2^* \subseteq \rightarrow_{12}^* \quad (*)$$

The relation $\rightarrow_1^* \circ \rightarrow_2^*$ has the diamond property:



With (*) and Lemma A.2.5 it immediately follows that \rightarrow_{12} is confluent.

Exercise 2 (Strong Confluence)

A relation \rightarrow is said to be *strongly confluent* iff:

$$t_2 \leftarrow s \rightarrow t_1 \implies \exists u. t_2 \rightarrow^* u \leftarrow^* t_1$$

Show that every *strongly confluent* relation is also *confluent*.

Solution

We show that every strongly confluent relation is also semi-confluent (see homework). To do so, we will show the stronger property

$$t_2 \xrightarrow{n} \leftarrow s \rightarrow t_1 \implies \exists u. t_2 \xrightarrow{=} u \xrightarrow{*} \leftarrow t_1$$

by induction on n . The base case for $n = 0$ is easy: $s \xrightarrow{0} \leftarrow s \rightarrow t_1 \implies s \xrightarrow{=} t_1 \xrightarrow{*} \leftarrow t_1$. For the induction step, we assume the statement for some u as the induction hypothesis. Furthermore, we assume $t'_2 \leftarrow t_2 \xrightarrow{n} \leftarrow s \rightarrow t_1$ for some t'_2 . We need to show that there exists a v with $t'_2 \xrightarrow{=} v \xrightarrow{*} \leftarrow t_1$.

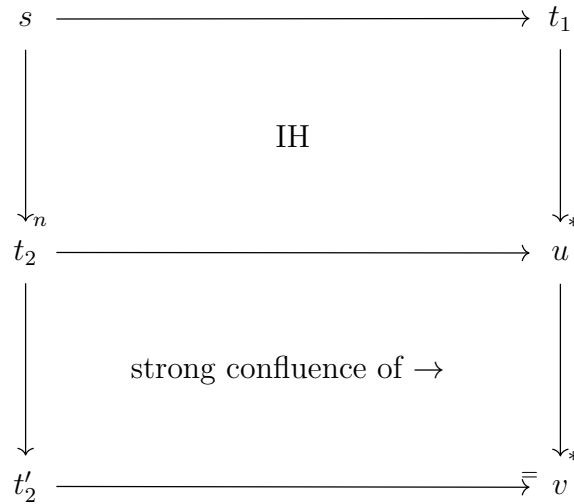
We make a case distinction on $t_2 \xrightarrow{=} u$.

When $t_2 = u$ we get $t_1 \xrightarrow{*} t_2$ with the IH and thus $t'_2 \xrightarrow{=} t'_2 \xrightarrow{*} \leftarrow t_1$ since $t_2 \rightarrow t'_2$.

If $t_2 \rightarrow u$, then from strong confluence of \rightarrow with $t'_2 \leftarrow t_2 \rightarrow u$ we obtain a v such that

$$u \xrightarrow{*} v \wedge t'_2 \xrightarrow{=} v$$

Together with the induction hypothesis $t_1 \xrightarrow{*} u$, we get $t'_2 \xrightarrow{=} v \xrightarrow{*} \leftarrow t_1$. As a picture:



Exercise 3 (Local Confluence of η -reduction)

Analogously to β -reduction, we define η -reduction inductively:

1. $x \notin \text{FV}(s) \implies (\lambda x. s x) \rightarrow_\eta s$
2. $s \rightarrow_\eta s' \implies s t \rightarrow_\eta s' t$
3. $s \rightarrow_\eta s' \implies t s \rightarrow_\eta t s'$
4. $s \rightarrow_\eta s' \implies (\lambda x. s) \rightarrow_\eta (\lambda x. s')$

The proof of local confluence of \rightarrow_η , i.e. it holds that there exists a u with $t_1 \xrightarrow{*}_\eta u \xrightarrow{*}_\eta t_2$ if we have $t_1 \xrightarrow{\eta} s \rightarrow_\eta t_2$, was very informal. Give a proper proof using this definition.

Solution

Proof by rule induction on $s \rightarrow_\eta t_1$. We only show the first, second, and last case. The third case is symmetric to the second one.

Case $s = (\lambda x. t_1 x) \rightarrow_\eta t_1$ **with** $x \notin \text{FV}(t_1)$:

By rule inversion on $(\lambda x. t_1 x) \rightarrow_\eta t_2$ we get the following two cases.

Case $(\lambda x. t_1 x) \rightarrow_\eta t_1 = t_2$: In this case we choose $u = t_1 = t_2$ and have $t_1 \rightarrow_\eta^* u$ as well as $t_2 \rightarrow_\eta^* u$ by reflexivity.

Case $(\lambda x. t_1 x) \rightarrow_\eta (\lambda x. t'_1 x) = t_2$ **where** $t_1 \rightarrow_\eta t'_1$: Then, we choose $u = t'_1$ and it holds that $t_1 \rightarrow_\eta t'_1$ as well as $t_2 = (\lambda x. t'_1 x) \rightarrow_\eta t'_1 = u$. The latter holds because the set of free variables is invariant under η -reduction.

Case $s = s_1 t_{12} \rightarrow_\eta t_{11} t_{12} = t_1$ **where** $s_1 \rightarrow_\eta t_{11}$:

We assume the I.H. that $s_1 \rightarrow_\eta s'_1 \implies \exists u. t_{11} \rightarrow_\eta^* u \xrightarrow{\eta^*} s'_1$ for arbitrary s'_1 . Again, we perform a case analysis on the derivation of $s = s_1 t_{12} \rightarrow_\eta t_2$.

Case $s = s_1 t_{12} \rightarrow_\eta s'_1 t_{12} = t_2$ **where** $s_1 \rightarrow_\eta s'_1$:

From the I.H. we obtain a u_1 with $t_{11} \rightarrow_\eta^* u_1$ and $s'_1 \rightarrow_\eta^* u_1$. Then, we have $t_1 = t_{11} t_{12} \rightarrow_\eta^* u_1 t_{12} = u$ and $t_2 = s_1 t_{12} \rightarrow_\eta s'_1 t_{12} \rightarrow_\eta^* u_1 t_{12} = u$. Here, we use the fact that we can η -reduce under application.

Case $s = s_1 t_{12} \rightarrow_\eta s_1 t'_{12} = t_2$ **where** $t_{12} \rightarrow_\eta t'_{12}$:

We have $t_1 = t_{11} t_{12} \rightarrow_\eta t_{11} t'_{12} = u$ and $t_2 = s_1 t'_{12} \rightarrow_\eta t_{11} t'_{12} = u$.

Case $s = (\lambda x. s_1) \rightarrow_\eta (\lambda x. s'_1) = t_1$ **where** $s_1 \rightarrow_\eta s'_1$:

We analyse the cases of the derivation $s = (\lambda x. s_1) \rightarrow_\eta t_2$.

Case $(\lambda x. s_1) \rightarrow_\eta (\lambda x. s''_1) = t_2$ **where** $s_1 \rightarrow_\eta s''_1$:

Then, we have $s_1 \rightarrow_\eta s'_1$ and $s_1 \rightarrow_\eta s''_1$ which, using the I.H., lets us obtain a u_1 with $s'_1 \rightarrow_\eta^* u_1$ and $s''_1 \rightarrow_\eta^* u_1$. Using the fact that we can η -reduce under the abstraction, we conclude that $t_1 = (\lambda x. s'_1) \rightarrow_\eta^* (\lambda x. u_1)$ and $t_2 = (\lambda x. s''_1) \rightarrow_\eta^* (\lambda x. u_1)$.

Case $(\lambda x. s_1) \rightarrow_\eta s_{11} = t_2$ **where** $s_1 = s_{11} x$ **and** $x \notin \text{FV}(s_{11})$:

There must exist s'_{11} such that $s'_1 = s'_{11} x$ and $s_{11} \rightarrow_\eta s'_{11}$. Additionally, we have that $x \notin \text{FV}(s'_{11})$. Thus, we get $t_1 = (\lambda x. s'_1) = (\lambda x. s'_{11} x) \rightarrow_\eta s'_{11}$ and $t_2 = s_{11} \rightarrow_\eta s'_{11}$.

Homework 4 (Semi-Confluence)

A relation \rightarrow is said to be *semi-confluent* iff:

$$t_2 \xrightarrow{*} s \rightarrow t_1 \implies \exists u. t_2 \xrightarrow{*} u \xrightarrow{*} t_1$$

Show that \rightarrow is *semi-confluent* if and only if it is *confluent*.

Homework 5 (Diamond Property & Normal Forms)

Show that if \rightarrow has the diamond property, every element is either in normal form or has no normal form.

Homework 6 (Weak Diamond Property)

Assume that \rightarrow has the following weaker diamond property:

$$t_2 \leftarrow s \rightarrow t_1 \wedge t_1 \neq t_2 \implies \exists u. t_2 \rightarrow u \leftarrow t_1.$$

- a) Is it still the case that every element is either in normal form or has no normal form?
- b) Show that if t has a normal form, then all its reductions to its normal form have the same length.