

**Exercise 1 ( $\beta$ -reduction on de Bruijn Preserves Substitution)**

We consider an alternative representation of  $\lambda$ -terms that is due to de Bruijn. In this representation,  $\lambda$ -terms are defined according to the following grammar:

$$d ::= i \in \mathbb{N}_0 \mid d_1 d_2 \mid \lambda d$$

Remember that we defined substitution for de Bruijn terms as follows:

$$i \uparrow_l = \begin{cases} i, & \text{if } i < l \\ i + 1, & \text{if } i \geq l \end{cases}$$

$$(d_1 d_2) \uparrow_l = d_1 \uparrow_l d_2 \uparrow_l$$

$$(\lambda d) \uparrow_l = \lambda d \uparrow_{l+1}$$

$$i[t/j] = \begin{cases} i & \text{if } i < j \\ t & \text{if } i = j \\ i - 1 & \text{if } i > j \end{cases}$$

$$(d_1 d_2)[t/j] = (d_1[t/j]) (d_2[t/j])$$

$$(\lambda d)[t/j] = \lambda (d[t \uparrow_0 / j + 1])$$

For the  $\beta$ -reduction, we only need to modify the case of the substitution. In particular, we define  $(\lambda d) e \rightarrow_\beta d[e/0]$ .

Prove that  $s[u/i] \rightarrow_\beta s'[u/i]$  if  $s \rightarrow_\beta s'$ .

**Exercise 2 (Strong Confluence)**

A relation  $\rightarrow$  is said to be *strongly confluent* iff:

$$t_2 \leftarrow s \rightarrow t_1 \implies \exists u. t_2 \rightarrow^* u \leftarrow^* t_1$$

Show that every *strongly confluent* relation is also *confluent*.

**Exercise 3 (Diamond Property & Normal Forms)**

Show that if  $\rightarrow$  has the diamond property, every element is either in normal form or has no normal form.

#### Homework 4 (Semi-Confluence)

A relation  $\rightarrow$  is said to be *semi-confluent* iff:

$$t_2 \stackrel{*}{\leftarrow} s \rightarrow t_1 \implies \exists u. t_2 \rightarrow^* u \stackrel{*}{\leftarrow} t_1$$

Show that  $\rightarrow$  is *semi-confluent* if and only if it is *confluent*.

#### Homework 5 (Weak Diamond Property)

Assume that  $\rightarrow$  has the following weaker diamond property:

$$t_2 \leftarrow s \rightarrow t_1 \wedge t_1 \neq t_2 \implies \exists u. t_2 \rightarrow u \leftarrow t_1.$$

- a) Is it still the case that every element is either in normal form or has no normal form?
- b) Show that if  $t$  has a normal form, then all its reductions to its normal form have the same length.