

# First-Order Logic Equality

# Predicate logic with equality

Predicate logic  
+  
distinguished predicate symbol “=” of arity 2

Semantics: A structure  $\mathcal{A}$  of predicate logic with equality always maps the predicate symbol = to the identity relation:

$$\mathcal{A}(=) = \{(d, d) \mid d \in U_{\mathcal{A}}\}$$

# Expressivity

## Fact

A structure is model of  $\exists x \forall y \ x=y$  iff its universe is a singleton.

## Theorem

Every satisfiable formula of predicate logic has a countably infinite model.

**Proof** Let  $F$  be satisfiable.

We assume w.l.o.g. that  $F = \forall x_1 \dots \forall x_n F^*$  and the variables occurring in  $F^*$  are exactly  $x_1, \dots, x_n$ .

(If necessary bring  $F$  into closed Skolem form).

We consider two cases:

$n = 0$ . **Exercise.**

$n > 0$ . Let  $G = \forall x_1 \dots \forall x_n F^*[f(x_1)/x_1]$ , where  $f$  is a function symbol that does not occur in  $F^*$ .  $G$  is satisfiable (**why?**) and  $T(G)$  is countably infinite. It follows from the fundamental theorem that  $G$  has a countably infinite model.

## Modelling equality

Let  $F$  be a formula of predicate logic with equality.

Let  $Eq$  be a predicate symbol that does not occur in  $F$ .

Let  $E_F$  be the conjunction of the following formulas:

$$\forall x \text{ Eq}(x, x)$$

$$\forall x \forall y (\text{Eq}(x, y) \rightarrow \text{Eq}(y, x))$$

$$\forall x \forall y \forall z ((\text{Eq}(x, y) \wedge \text{Eq}(y, z)) \rightarrow \text{Eq}(x, z))$$

For every function symbol  $f$  in  $F$  of arity  $n$  and every  $1 \leq i \leq n$ :

$$\forall x_1 \dots \forall x_n \forall y (\text{Eq}(x_i, y) \rightarrow \\ \text{Eq}(f(x_1, \dots, x_i, \dots, x_n), f(x_1, \dots, y, \dots, x_n)))$$

For every predicate symbol  $P$  in  $F$  of arity  $n$  and every  $1 \leq i \leq n$ :

$$\forall x_1 \dots \forall x_n \forall y (\text{Eq}(x_i, y) \rightarrow \\ (P(x_1, \dots, x_i, \dots, x_n) \leftrightarrow P(x_1, \dots, y, \dots, x_n)))$$

$E_F$  expresses that  $Eq$  is a *congruence relation* on the symbols in  $F$ .

# Quotient structure

## Definition

Let  $\mathcal{A}$  be a structure and  $\sim$  an equivalence relation on  $U_{\mathcal{A}}$  that is a congruence relation for all the predicate and function symbols defined by  $I_{\mathcal{A}}$ . The **quotient structure**  $\mathcal{A}/\sim$  is defined as follows:

- ▶  $U_{\mathcal{A}/\sim} = \{[u]_{\sim} \mid u \in U_{\mathcal{A}}\}$  where  $[u]_{\sim} = \{v \in U_{\mathcal{A}} \mid u \sim v\}$
- ▶ For every function symbol  $f$  defined by  $I_{\mathcal{A}}$ :  
 $f^{\mathcal{A}/\sim}([d_1]_{\sim}, \dots, [d_n]_{\sim}) = [f^{\mathcal{A}}(d_1, \dots, d_n)]_{\sim}$
- ▶ For every predicate symbol  $P$  defined by  $I_{\mathcal{A}}$ :  
 $P^{\mathcal{A}/\sim}([d_1]_{\sim}, \dots, [d_n]_{\sim}) = P^{\mathcal{A}}(d_1, \dots, d_n)$
- ▶ For every variable  $x$  defined by  $I_{\mathcal{A}}$ :  $x^{\mathcal{A}/\sim} = [x^{\mathcal{A}}]_{\sim}$

## Lemma

$$\mathcal{A}/\sim(t) = [\mathcal{A}(t)]_{\sim}$$

## Lemma

$$\mathcal{A}/\sim(F) = \mathcal{A}(F)$$

## Theorem

*The formulas  $F$  and  $E_F \wedge F[Eq/=]$  are equisatisfiable.*

**Proof** We show that if  $E_F \wedge F[Eq/=]$  is sat., then  $F$  is satisfiable.

Assume  $\mathcal{A} \models E_F \wedge F[Eq/=]$ .

$\Rightarrow Eq^{\mathcal{A}}$  is an congruence relation.

Let  $\mathcal{B} = \mathcal{A}/_{Eq^{\mathcal{A}}}$  (extended with  $=$  interpreted as identity).

$\Rightarrow \mathcal{B} \models F[Eq/=]$

By construction  $Eq^{\mathcal{B}}$  is identity:

$$Eq^{\mathcal{B}}([a], [a']) = Eq^{\mathcal{A}}(a, a') = ([a]_{Eq^{\mathcal{A}}} = [a']_{Eq^{\mathcal{A}}})$$

$\Rightarrow \mathcal{B}(F[Eq/=]) = \mathcal{B}(F)$

$\Rightarrow \mathcal{B} \models F$