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Exercise 3.1. [**System G1c**]
An alternative definition of the sequent calculus ("G1c") is defined as follows, where A, B are formulas:

Axioms

$$\text{Ax}\ \ A \Rightarrow A \qquad\qquad \text{L}\bot\ \ \bot \Rightarrow$$

Rules for weakening (W) and contraction (C)

$$\text{LW}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad\qquad \text{RW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

$$\text{LC}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad\qquad \text{RC}\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

Rules for the logical operators

$$\text{L}\wedge\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta}\ (i = 0, 1) \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \text{R}\vee\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1}\ (i = 0, 1)$$

$$\text{L}{\to}\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \text{R}{\to}\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

Notably, weaking and contraction are built-in rules. Moreover, for system G1c, we define $\neg F \coloneqq F \to \bot$.

Show that sequent calculus can be simulated by G1c, i.e., $\vdash_G \Gamma \Rightarrow \Delta$ implies $\vdash_{G1c} \Gamma \Rightarrow \Delta$.

Exercise 3.2. [**Cut Elimination, Semantically**]
Semantically prove the admissibility of the following rule:

$$\text{If } \vdash_G \Gamma \Rightarrow F, \Delta \text{ and } \vdash_G F, \Gamma \Rightarrow \Delta \text{ then } \vdash_G \Gamma \Rightarrow \Delta$$

Exercise 3.3. [**More Connectives**]
Define sequent rules for the logical connectives "nand" ($\overline{\wedge}$) and "xor" ($\otimes$).

Exercise 3.4. [**Inversion**]
Show that the following inversion rule is admissible using proof transformations:

$$\frac{\Gamma \Rightarrow F \vee G, \Delta}{\Gamma \Rightarrow F, G, \Delta}$$

Homework 3.1. [Stay Classy] (++)

1. Prove the formulas $F \vee \neg F$ (tertium non datur) and $(\neg F \to F) \to F$ (consequentia mirabilis) in System G1c.
2. The intuitionistic system "G1l" is the subsystem of G1c obtained by restricting all rules to sequents with at most one succedent formula and by replacing the rule $L{\to}$ with

$$\frac{\Gamma \Rightarrow F \qquad G, \Gamma \Rightarrow H}{F \to G, \Gamma \Rightarrow H}$$

   Can you prove either of former formulas in G1l? How about their doubly negated forms $\neg\neg(F \vee \neg F)$ and $\neg\neg((\neg F \to F) \to F)$?

Homework 3.2. [Inversion Rules] (++)
Show that the following inversion rules are admissible using proof transformations:

$$\frac{F \wedge G, \Gamma \Rightarrow \Delta}{F, G, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow F \to G, \Delta}{F, \Gamma \Rightarrow G, \Delta}$$

Homework 3.3. [Sequent Prover] (+++)
Implement a sequent calculus prover in a high-level programming language, and test it for examples from this exercise sheet, the lecture, or your own.

> If I had a world of my own, everything would be nonsense.
>
> — Alice