

LOGIC EXERCISES

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EXERCISE SHEET 5

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Exercise 5.1. [Induction Proofs]

Show that the following schema has a proof in natural deduction for all $n \geq 1$:

$$P_n = ((A_1 \wedge (A_2 \wedge (\dots \wedge A_n) \dots)) \rightarrow B) \rightarrow (A_1 \rightarrow (A_2 \rightarrow (\dots (A_n \rightarrow B) \dots)))$$

Solution:

- Case $n = 1$:
Formula: $P_1 = (A_1 \rightarrow B) \rightarrow (A_1 \rightarrow B)$
Proof: trivial using ($\rightarrow I$)
- Case $n + 1$:

$$\begin{array}{c}
 \text{(IH)} \frac{\frac{\frac{\frac{\overline{\overline{B} \text{ (subproof)}}}{A_2 \wedge \dots \wedge A_{n+1} \rightarrow B} \rightarrow I_3}{A_2 \wedge \dots \wedge A_{n+1} \rightarrow B} \rightarrow E}{A_2 \rightarrow \dots \rightarrow A_{n+1} \rightarrow B} \rightarrow I_2}{A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_{n+1} \rightarrow B} \rightarrow I_1}{(A_1 \wedge A_2 \wedge \dots \wedge A_{n+1} \rightarrow B) \rightarrow (A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_{n+1} \rightarrow B)} \rightarrow I_1
 \end{array}$$

Subproof:

$$\frac{[A_1 \wedge A_2 \wedge \dots \wedge A_{n+1} \rightarrow B]^1 \quad \frac{[A_1]^2 \quad [A_2 \wedge \dots \wedge A_{n+1}]^3}{A_1 \wedge A_2 \wedge \dots \wedge A_{n+1}} \wedge I}{B} \rightarrow E$$

Exercise 5.2. [From Sequent Calculus to Natural Deduction]

How can we construct a natural deduction proof $\Gamma \vdash_N \bigvee \Delta$ from a sequent calculus proof $\Gamma \Rightarrow \Delta$?

Solution:

1. Known from lecture: If $\Gamma \Rightarrow \Delta$ then $\Gamma, \neg\Delta \vdash_N \perp$
2. If $\Gamma, \neg\Delta \vdash_N \perp$ then $\Gamma, \neg\bigvee \Delta \vdash_N \perp$ (to be shown)
3. Use the (\perp) rule to go from $\Gamma, \neg\bigvee \Delta \vdash_N \perp$ to $\Gamma \vdash_N \bigvee \Delta$

Proof of 2.: First we show that $\neg\bigvee \Delta \vdash \neg F_i$ for any $F_i \in \Delta$. For simplicity, we focus on the binary case $\Delta = \{F, G\}$. We have:

$$\frac{\frac{[\neg(F \vee G)] \quad \frac{[F]^1}{F \vee G}(\vee E_1)}{\perp}(\rightarrow E)}{\neg F}(\rightarrow I)_1$$

and similar for $\neg G$.

Now given a proof of $\Gamma, \neg\Delta \vdash_N \perp$, we replace all closed assumptions $[\neg F_i]$ for $F_i \in \Delta$ in the proof by a proof of $\Gamma, \neg\bigvee \Delta \vdash_N \neg F_i$ as constructed in the previous step. This gives us a proof of $\Gamma, \neg\bigvee \Delta \vdash_N \perp$

Exercise 5.3. [Hilbert Calculus]

Prove the following formula with a linear proof in Hilbert calculus: $(F \wedge G) \rightarrow (G \wedge F)$

Hint: Use the deduction theorem.

Solution:

We first apply the deduction theorem. It remains to construct a proof of $F \wedge G \vdash_H G \wedge F$.

- | | |
|---|----------|
| 1. $F \wedge G \rightarrow G$ | A5 |
| 2. $F \wedge G$ | Γ |
| 3. $F \wedge G \rightarrow F$ | A4 |
| 4. $G \rightarrow F \rightarrow G \wedge F$ | A3 |
| 5. G | 1, 2 |
| 6. $F \rightarrow G \wedge F$ | 4, 5 |
| 7. F | 3, 2 |
| 8. $G \wedge F$ | 6, 7 |

Homework 5.1. [Small Hilbert] (++)

In the lecture, Hilbert calculus for propositional logic was introduced by means of nine axioms. However, the following three axioms are already sufficient:

$$\mathbf{A1} \quad F \rightarrow (G \rightarrow F)$$

$$\mathbf{A2} \quad (F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$$

$$\mathbf{A10} \quad (\neg F \rightarrow \neg G) \rightarrow (G \rightarrow F)$$

Derive the following statement from the axioms above with the help of \rightarrow_E :

$$\neg(F \rightarrow F) \rightarrow G$$

Bonus: In fact, **Meredith** showed that all that is needed is one single axiom:

$$((((A \rightarrow B) \rightarrow (\neg C \rightarrow \neg D)) \rightarrow C) \rightarrow E) \rightarrow ((E \rightarrow A) \rightarrow (D \rightarrow A))$$

Try to derive some axiom of your choice presented in the lecture in Meredith's system.

Solution:

If you got stuck, reach out on Zulip.

Homework 5.2. [From Sequent Calculus to Natural Deduction: Reloaded] (++)

In Exercise 5.2, we constructed a natural deduction proof $\Gamma \vdash_N \bigvee \Delta$ from a sequent calculus proof of $\Gamma \Rightarrow \Delta$. That construction created classical proofs because it required the use of the (\perp) rule.

Let us consider yet another restricted Sequent Calculus called "G3c". In G3c, we have $\Delta = \{F\}$, that is the succedent always contains exactly one formula. Here are the axioms:

$$\begin{array}{ll} \mathbf{Ax} \quad P, \Gamma \Rightarrow P \text{ (} P \text{ atomic)} & \mathbf{L\perp} \quad \perp, \Gamma \Rightarrow A \\ \mathbf{L\wedge} \quad \frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} & \mathbf{R\wedge} \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \\ \mathbf{L\vee} \quad \frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} & \mathbf{R\vee} \quad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1} \text{ (} i = 0, 1 \text{)} \\ \mathbf{L\rightarrow} \quad \frac{A \rightarrow B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \rightarrow B, \Gamma \Rightarrow C} & \mathbf{R\rightarrow} \quad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \end{array}$$

Give a direct construction that transforms a G3c proof $\Gamma \Rightarrow F$ into a natural deduction proof $\Gamma \vdash_N F$ without using the (\perp) rule. You are, however, allowed to use the intuitionistic rule $\perp \vdash_N F$; call it $(\perp E)$.

Solution:

Case $\perp\perp$: Follows by using ($\perp E$).

Case $\perp A$:

$$\frac{\frac{C}{A \rightarrow C} (\rightarrow I)_2 \quad \frac{[A \wedge B]}{A} (\wedge E_1)}{C} \quad \frac{\frac{C}{B \rightarrow C} (\rightarrow I)_1 \quad \frac{[A \wedge B]}{B} (\wedge E_2)}{C} (\rightarrow E)$$

Case $R\perp$: Follows from ($\perp I$) + IH

$$\frac{[A \vee B] \quad \frac{[A]}{C} (IH) \quad \frac{[B]}{C} (IH)}{C} (VE)$$

Case $R\vee$: (IH) + ($\vee I_i$), $i \in \{1, 2\}$

Case $L\rightarrow$:

$$\frac{\frac{C}{B \rightarrow C} (\rightarrow I)_1 \quad \frac{[A \rightarrow B]}{B} (\rightarrow E)}{C} \quad \frac{\text{of first premise} \downarrow (IH) \quad A}{A} (\rightarrow E)$$

Case $R\rightarrow$: (IH) + ($\rightarrow I$).

□

Homework 5.3. [Simulating Truth Tables] (+)

In the lecture, the following lemma was discussed:

Let $\text{atoms}(F) \subseteq \{A_1, \dots, A_n\}$. Then we can construct a proof $A_1^A, \dots, A_n^A \vdash_N F^A$.

Recall the definition of F^A :

$$F^A = \begin{cases} F, & \text{if } \mathcal{A}(F) = 1 \\ \neg F, & \text{otherwise} \end{cases}$$

The proof proceeded by induction on F . The cases for atomic formulas as well as implication were shown in the lecture. Prove the cases for negation and disjunction!

Solution:

Case $F \equiv \neg F'$: Assume $\mathcal{A}(F) = 1$. Then $\mathcal{A}(F') = 0$. Hence, $F^A = F = \neg F' = F'^A$. By the IH, we have a proof of $A_1^A, \dots, A_n^A \vdash_N F'^A$, which finishes the case.

Assume $\mathcal{A}(F) = 0$. Then $\mathcal{A}(F') = 1$. Hence, $F^A = \neg F = \neg \neg F'$ and $F'^A = F'$. By the IH, we have a proof of $A_1^A, \dots, A_n^A \vdash_N F'^A$. To conclude, we build the following proof tree:

$$\frac{\frac{[\neg F']^1 \quad \overline{F'}(IH)}{\perp} (\rightarrow E)}{\neg \neg F'} (\rightarrow I)_1$$

Case $F \vee G$: Assume $\mathcal{A}(F) = 1$. Then $\mathcal{A}(F) = 1$ or $\mathcal{A}(G) = 1$. In case of the former, we build

$$\frac{\overline{F}(IH)}{F \vee G} (\vee I_1)$$

Note that $\text{atoms}(F \vee G) = \text{atoms}(F) \cup \text{atoms}(G)$, which we used in order to apply the IH. The other case is symmetric.

Assume $\mathcal{A}(F \vee G) = 0$. Then $\mathcal{A}(F) = 0$ and $\mathcal{A}(G) = 0$. By the IH, we obtain proofs

$$\begin{aligned} A_1^A, \dots, A_n^A &\vdash_N \neg F \\ A_1^A, \dots, A_n^A &\vdash_N \neg G \end{aligned}$$

To conclude, we build the following proof tree:

$$\begin{array}{c}
 \frac{[\neg F \wedge \neg G]^2}{\neg F} (\wedge E_1) \quad [F]^3 \quad \vdots}{\perp} (\rightarrow E) \quad \frac{\vdots}{\perp} (\rightarrow E) \\
 \hline
 \frac{\perp}{\neg F \wedge \neg G \rightarrow \perp} (\vee E)_3 \quad \frac{\perp}{\neg F \wedge \neg G \rightarrow \perp} (\rightarrow I)_2 \quad \frac{\overline{\neg F} (IH) \quad \overline{\neg G} (IH)}{\neg F \wedge \neg G} (\wedge I) \\
 \hline
 \frac{\perp}{\neg(F \vee G)} (\rightarrow E) \quad \frac{\perp}{\neg(F \vee G)} (\rightarrow I)_1
 \end{array}$$

Simplicity is the ultimate sophistication.

— William Gaddis