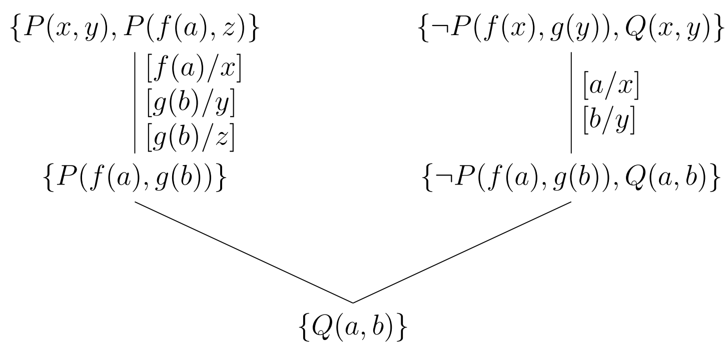
Exercise 9.1. [Wait, What?]

1. Resolution for first-order logic is sound and complete.
2. The satisfiability and validity problems for first-order logic are undecidable.

How do you reconcile these two facts? Write down the definitions of all above used logical terminology (sound, complete, undecidable, etc.) and discuss the consequences of above facts.

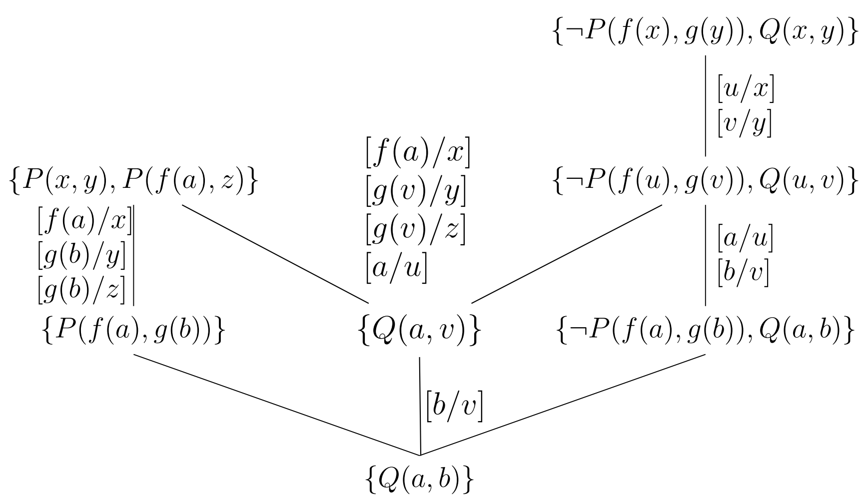
Solution:
Definitions: cf lecture; resolution is just a semi-decision procedure: if a formula is satisfiable, resolution might not terminate.

Exercise 9.2. [Do You Even Lifting Lemma?]
Consider the following resolution:



Follow the proof of the Lifting Lemma and find out which (predicate logic) resolution step is constructed from this.

Solution:
The missing predicate resolution step can be depicted as follows:



Exercise 9.3. [Green Dragon Children Are Cute Unless You Have to Fight Them]
Express the following facts by formulas in predicate logic.

1. Every dragon is happy if all its children can fly.
2. Green dragons can fly.
3. A dragon is green if it is a child of at least one green dragon.

Prove by resolution that the conjunction of these three statements implies the following: all green dragons are happy.

Solution:
We use unary predicates H, G and F to describe that a dragon is happy, green, and it can fly, respectively, and a binary predicate C to describe a dragon being a child of another dragon. Then the sentences in English can be expressed as follows:

1. $F_1 = \forall x \left(\forall y \ (C(y,x) \to F(y)) \to H(x)\right)$
2. $F_2 = \forall x \ (G(x) \to F(x))$
3. $F_3 = \forall x \left(\exists y \ (C(x,y) \land G(y)) \to G(x)\right)$
4. $F_4 = \forall x \ (G(x) \to H(x))$

We need to prove that the last formula is entailed by the previous three, formally, $F_1 \land F_2 \land F_3 \models F_4$. Equivalently, we prove that $F_1 \land F_2 \land F_3 \land \neg F_4$ is unsatisfiable.

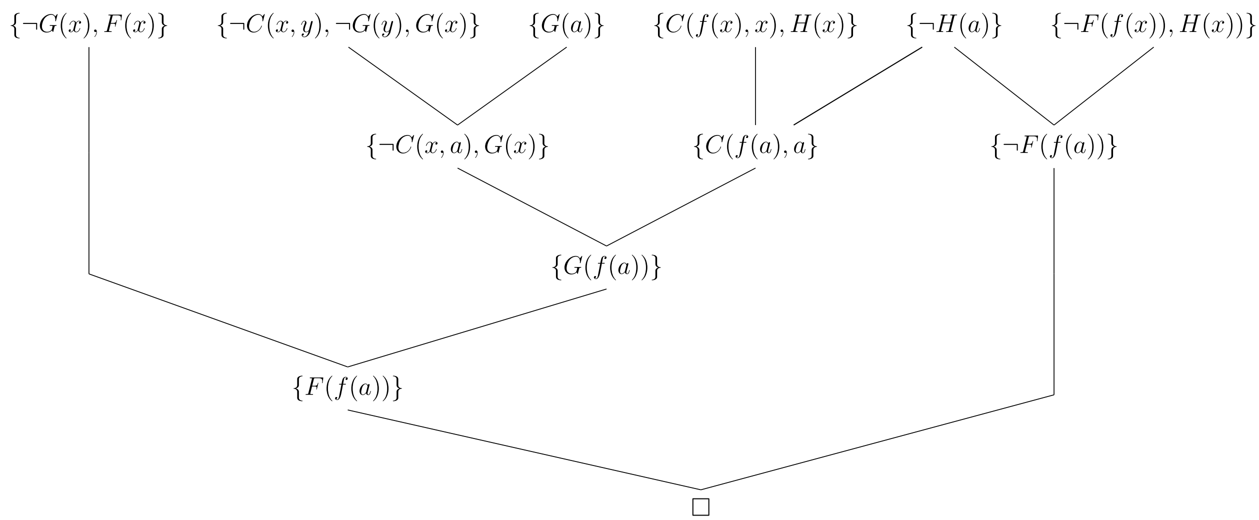
First we transform each formula into the required Skolem form with matrices in CNF:

$$\begin{aligned} F_1 &\equiv \forall x \exists y \left((C(y,x) \lor H(x)) \land (\neg F(y) \lor H(x))\right) \\ &\equiv_s \forall x \left((C(f(x),x) \lor H(x)) \land (\neg F(f(x)) \lor H(x))\right) \end{aligned}$$

$$F_2 \equiv \forall x \ (\neg G(x) \lor F(x))$$
$$F_3 \equiv \forall x \forall y \left(\neg C(x,y) \lor \neg G(y) \lor G(x)\right)$$

$$\begin{aligned} \neg F_4 &\equiv \exists x \left(G(x) \land \neg H(x)\right) \\ &\equiv_s G(a) \land \neg H(a) \end{aligned}$$

Finally, we create a resolution proof of the empty clause:



Exercise 9.4. **[Justice > Equity > Equality]**
We consider how to model equality in predicate logic. In the lecture slides, the following axiom schema for congruence is used:

$$\frac{Eq(x_i, y)}{Eq(f(x_1, \dots, x_i, \dots, x_n), f(x_1, \dots, y, \dots, x_n))}$$

Assume that this schema is replaced by:

$$\frac{Eq(x_1, y_1) \qquad \cdots \qquad Eq(x_n, y_n)}{Eq(f(x_1, \dots, x_n), f(y_1, \dots, y_n))}$$

Reflexivity, symmetry and transitivity stay unchanged. Show that the above modified schemas is equivalent to the schemas from the lecture.

Solution:
We first simulate the modified schema with the original one. Because the original schema only allows us to replace one term at a time, an induction is necessary. We want to prove $Eq(f(x_1, \dots, x_n), f(y_1, \dots, y_m, x_{m+1}, \dots, x_n))$ for $1 \leq m \leq n$. With $m = n$ we obtain the desired schema, hence the induction must proceed on m.

- Base case: $m = 1$

$$\frac{Eq(x_1, y_1)}{Eq(f(x_1, \dots, x_n), f(y_1, x_2, \dots, x_n))}$$

- Induction step: $m + 1$

$$\textsc{Trans}\ \frac{\text{IH}\ \dfrac{}{Eq(f(x_1, \dots, x_n), f(y_1, \dots, y_m, x_{m+1}, \dots, x_n))} \qquad \dfrac{Eq(x_{m+1}, y_{m+1})}{Eq(f(y_1, \dots, y_m, x_{m+1}, \dots, x_n), f(y_1, \dots, y_{m+1}, x_{m+2}, \dots, x_n))}}{Eq(f(x_1, \dots, x_n), f(y_1, \dots, y_{m+1}, x_{m+2}, \dots, x_n))}$$

Now, the opposite direction.

$$\frac{\dfrac{}{Eq(x_1, x_1)} \qquad \cdots \qquad Eq(x_i, y) \qquad \cdots \qquad \dfrac{}{Eq(x_n, x_n)}}{Eq(f(x_1, \dots, x_i, \dots, x_n), f(x_1, \dots, y, \dots, x_n))}$$

Homework 9.1. [Blackbox Proving] (++)

Assume you are given an algorithm $\mathcal{A}$ operating on first-order CNF formulas such that whenever the resolution calculus produces a resolvent R from clauses C_1 and C_2, $\mathcal{A}$ produces a clause $R' \subseteq R$.

Prove that $\mathcal{A}$ is refutationally complete.

Solution:

We prove a generalisation: whenever the resolution calculus proves $\square$ from a set of clauses S then $\mathcal{A}$ can prove $\square$ from any set S' satisfying $\forall C \in S. \exists C' \in S'. \exists \sigma. C'\sigma \subseteq C$. The proof is by induction on the number of resolution steps n:

In case $n = 0$, $\square \in S$ and hence $\square \in S'$ and we are done. In case $n + 1$, assume we are resolving two clauses C_1, C_2 on $L_1, \dots, L_m, L'_1, \dots, L'_n$ using σ to obtain R. Then there are $D_1, D_2 \in S'$ and σ_1, σ_2 with $D_1\sigma_1 \subseteq C_1$ and $D_2\sigma_2 \subseteq C_2$. WLOG $\mathsf{dom}(\sigma_i) = \mathsf{vars}(D_i)$ (otherwise consider the restriction of σ_i). Let $N_1 \coloneqq \{L \in D_1 \mid L\sigma_1 \in \{L_1, \dots, L_m\}\}$ and $N_2 \coloneqq \{L \in D_2 \mid L\sigma_2 \in \{L'_1, \dots, L'_n\}\}$.

If $N_1 \neq \emptyset \neq N_2$, then we can resolve D_1 and D_2 on N_1 and N_2 using $(\sigma_1 \cup \sigma_2)\sigma$ and thus $\mathcal{A}$ hands us $R' \subseteq R$. Now $S' \cup \{R'\}$ satisfies the invariant with respect to $S \cup \{R\}$ and thus we can apply the inductive hypothesis to conclude.

Assume one of N_1, N_2 is empty. WLOG assume it is N_1. We have $R \supseteq (C_1 \setminus \{L_1, \dots, L_m\})\sigma$. So $D_1\sigma_1\sigma \subseteq R$ and hence S' satisfies the invariant with respect to $S \cup \{R\}$. Thus we can apply the inductive hypothesis to conclude.

Homework 9.2. **[Restricted Resolution]** (+++)
In the resolution procedure as defined in the lecture slides, we can unify arbitrarily many literals from two clauses. Consider a modified resolution procedure where exactly one literal is picked in each clause ("binary resolution"). We add a new rule ("factoring"): for a clause $C = \{L_1, \dots, L_n\}$, where $\{L_i, L_j\}$ can be unified using an mgu σ with $i \neq j$, add another clause $C' = (C \setminus L_i)\sigma$.

For example, given the clause

$$C = \{\neg W(x), \neg W(f(y)), T(x,y), \neg W(f(c))\}$$

we can apply the factoring rule as follows:

$$L_1 = \neg W(x), L_2 = \neg W(f(y)), \sigma = \{x \mapsto f(y)\}, C' = \{\neg W(f(y)), T(f(y), y), \neg W(f(c))\}$$

1. Prove that restricted resolution without factoring is incomplete.
2. Prove that restricted resolution is complete.

Solution:
To see why factoring is necessary, consider the contradictory clauses $\{P(x), P(y)\}$ and $\{\neg P(v), \neg P(w)\}$ (try to derive the empty clause and explain why it fails!).

As for the completeness proof, it suffices to simulate the resolution procedure from the lecture. Assume the calculus produces a resolvent $R = \big((C_1 \setminus \{L_1, \dots, L_m\}) \cup (C_2\rho \setminus \{L'_1, \dots, L'_n\})\big)\sigma$. We show that we can derive $R' \subseteq R$ using our restricted resolution calculus. This then implies completeness by the previous homework exercise. We proceed by induction on $k := \max\{m, n\}$. If $k = 1$, then we can just use the binary resolution rule.

Assume $k > 1$ and WLOG $k = m$. By assumption, there is an mgu σ of $\{\overline{L_1}, \dots, \overline{L_m}, L'_1, \dots, L'_m\}$. WLOG assume $\mathsf{dom}(\sigma) \cap \mathsf{vars}((C_1 \cup C_2\rho)\sigma) = \emptyset$ (otherwise, obtain such an mgu from σ by renaming). By factoring, we obtain $C_3 := (C_1 \setminus \{L_1\})\sigma$.

Now let $\sigma' := \sigma_{|\mathsf{vars}(C_2\rho)}$ be the restriction of σ on $\mathsf{vars}(C_2\rho)$. Note that σ' is an mgu of $\{\overline{L_2\sigma}, \dots, \overline{L_m\sigma}, L'_1, \dots, L'_m\}$ (why?). So we can build the resolvent

$$R' = \big((C_3 \setminus \{L_2\sigma, \dots, L_m\sigma\}) \cup (C_2\rho \setminus \{L'_1, \dots, L'_n\})\big)\sigma'$$

using the original resolution calculus and hence, by the inductive hypothesis, also the resolvent $R'' \subseteq R'$ using our restricted resolution calculus. Finally note that

$$\begin{aligned}
R' &= \big((C_3 \setminus \{L_2\sigma, \dots, L_m\sigma\}) \cup (C_2\rho \setminus \{L'_1, \dots, L'_n\})\big)\sigma' \\
&= \big(((C_1 \setminus \{L_1\})\sigma \setminus \{L_2, \dots, L_m\}\sigma) \cup (C_2\rho \setminus \{L'_1, \dots, L'_n\})\big)\sigma' \\
&= \big((C_1\sigma \setminus \{L_1, \dots, L_m\}\sigma) \cup (C_2\rho \setminus \{L'_1, \dots, L'_n\})\big)\sigma' \\
&\subseteq \big((C_1 \setminus \{L_1, \dots, L_m\})\sigma \cup (C_2\rho \setminus \{L'_1, \dots, L'_n\})\big)\sigma' \\
&= \big((C_1 \setminus \{L_1, \dots, L_m\}) \cup (C_2\rho \setminus \{L'_1, \dots, L'_n\})\big)\sigma.
\end{aligned}$$

(check these steps carefully)

Homework 9.3. **[Equality Elimination]** (+)
Show with resolution that: $f(f(f(a))) = a \to \big(f(f(a)) = a \to f(a) = a\big)$ is valid. First, remove equality based on the procedure from the lecture. Then perform resolution.

Solution:
We obtain

$$Eq(f(f(f(a)), a) \to \big(Eq(f(f(a)), a) \to Eq(f(a) = a)\big).$$

Then negate and clausify

$$\{Eq(f(f(f(a)), a)\}, \{Eq(f(f(a)), a)\}, \{\neg Eq(f(a), a)\}.$$

Then add all clausified equality axioms:

$$\begin{aligned}\{Eq(f(f(f(a)), a)\}, \{Eq(f(f(a)), a)\}, \{\neg Eq(f(a), a)\}\\ \{Eq(x, x)\}, \{\neg Eq(x, y), Eq(y, x)\}, \{\neg Eq(x, y), \neg Eq(y, z), Eq(x, z)\}\\ \{\neg Eq(x, y), Eq(f(x), f(y))\}.\end{aligned}$$

Finally, resolve!

1. Resolving clause 2 and f axiom with $[f(f(a))/x, a/y]$: $\{Eq(f(f(f(a))), f(a))\}$
2. Resolving this clause and the symmetry axiom with $[f(f(f(a)))/x, f(a)/y]$: $\{Eq(f(a), f(f(f(a))))\}$.
3. Resolving this clause and the transitivity axiom with $[f(f(f(a)))/y, f(a)/x]$: $\{\neg Eq(f(f(f(a))), z), Eq(f(a), z)\}$
4. Resolving this clause and clause 3 with $[a/z]$: $\{\neg Eq(f(f(f(a))), a)\}$
5. Resolving this clause and clause 1: $\square$

Homework 9.4. **[(Bonus) A Barbarian Bavarian Barber Walks Into a Barber]** (++)

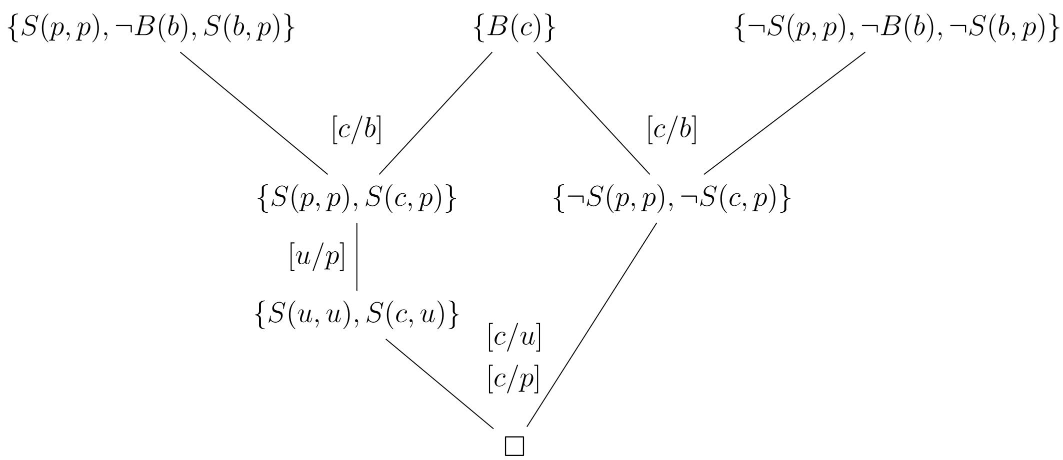
You can solve this exercise if you need more practice with FOL-resolution; but you will not miss anything if you do not – there is no new content in this exercise.

Consider the signature $\{B, S\}$ where B is a unary predicate expressing that an element represents a barber; while $S(x, y)$ indicates that "x shaves y".

1. Axiomatize the statements:
   - Persons who do not shave themselves are shaven by all barbers.
   - No barber shaves persons who shave themselves.
2. Show by resolution that the fact that no barbers exist is a consequence of the two statements above.

Solution:

1. - $\forall p \forall b \, . \, (\neg S(p,p) \wedge B(b) \rightarrow S(b,p))$
   - $\forall p \forall b \, . \, (S(p,p) \wedge B(b) \rightarrow \neg S(b,p)$
2. We obtain the clause set $\{\{S(p,p), \neg B(b), S(b,p)\}, \{\neg S(p,p), \neg B(b), \neg S(b,p)\}\}$.
   We show the entailment of $\varphi := \forall b \, . \, \neg B(b)$ by assuming $\neg\varphi$ and proving a contradiction. To this end, we skolemize $\neg \forall b \, . \, \neg B(b) \equiv \exists b \, . \, B(b)$ to $B(c)$.



   This proves the non-existence of barbers under the given assumptions. Behind what might appear as smoke and mirrors, the conclusion of this is actually easy to obtain by thinking about the shaving of barbers: Barbers shave themselves if and only if they do not shave themselves, which is the central contradiction to uncover.

> The past was erased,
> the erasure was forgotten,
> the lie became the truth.
>
> — George Orwell