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Chair for Logic and Verification

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SS 2021 | Exercise Sheet 11 | 25.06.2021

The tutorials on 02.07 and 09.07 have to be moved. Please participate in the poll on Zulip until 28.06 to find a new date for the tutorials.

Exercise 11.1. **[Decidability of Complete Theories]**
Assume S is finitely axiomatizable and complete, i.e. $F \in S$ or $\neg F \in S$ for any sentence F.

1. Given only the axiomatization of S, give a procedure deciding whether $S \models F$ for any sentence F.
2. Can you obtain a similar result when the assumption is that the axiom system is only *recursively enumerable*?

Solution:

1. Let M be the set of axioms. Run resolution on $M \wedge F$ and $M \wedge \neg F$ in parallel. If $F \notin S$, then $M \wedge F \vdash \square$ and the first resolution terminates. If $F \in S$, then $M \wedge \neg F \vdash \square$ and the second resolution terminates.
2. Yes, by compactness. Enumerate all finite subsets of the axiom set and execute the resolution calls in a dovetailing approach.

Exercise 11.2. **[Consequence]**
Show that Cn is a closure operator, i.e. Cn fulfills the following properties:

- $S \subseteq Cn(S)$
- if $S \subseteq S'$ then $Cn(S) \subseteq Cn(S')$
- $Cn(Cn(S)) = Cn(S)$

Solution:
In the following, suppose S, S' are sets of Σ-sentences and F is a Σ-sentence.

- $F \in S \Longrightarrow S \models F \Longrightarrow F \in Cn(S)$
- $F \in Cn(S) \Longrightarrow S \models F \Longrightarrow S' \models F \Longrightarrow F \in Cn(S')$
- From the first property, we get $Cn(S) \subseteq Cn(Cn(S))$. For the other direction, we have $F \in Cn(Cn(S)) \Longrightarrow Cn(S) \models F \Longrightarrow^{(*)} S \models F \Longrightarrow F \in Cn(S)$.
  We have (*) because $\mathcal{A} \models Cn(S)$ iff $\mathcal{A} \models S$ by definition of Cn.

Exercise 11.3. **[One Finite, All Finite]**
Show that if a theory is finitely axiomatizable, any countable axiomatization of it has a finite subset that axiomatizes the same theory. In other words, if $Cn(\Gamma) = Cn(\Delta)$ with Γ countable and Δ finite, then there is a finite $\Gamma' \subseteq \Gamma$ with $Cn(\Gamma') = Cn(\Gamma)$. Can you also obtain Γ' effectively?

Solution:
Let us identify Δ as the formula $\bigwedge_{F\in\Delta} F$. It suffices to find a finite subset $\Gamma' \subseteq \Gamma$ that axiomatizes $Cn(\Delta)$. For this, it is sufficient to find $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models \Delta$, which is equivalent to $\Gamma' \cup \{\neg\Delta\}$ being unsatisfiable.

We know that $\Gamma \cup \{\neg\Delta\}$ is unsatisfiable because Γ axiomatizes $Cn(\Delta)$. By compactness, there must be a finite subset that is unsatisfiable. We can find this subset by enumerating all finite subsets $\Gamma' \subseteq \Gamma$ and running resolution on $\Gamma', \neg\Delta$.

Exercise 11.4. **[Natural Deduction]**
Prove the following formula using natural deduction.

$$\neg(\forall x(\exists y(\neg P(x) \wedge P(y))))$$

Solution:

$$
\dfrac{\dfrac{
  \forall E\ \dfrac{[\forall x \exists y(\neg P(x)\wedge P(y))]}{\exists y(\neg P(x_1)\wedge P(y))}
  \qquad
  \dfrac{
    \dfrac{[\neg P(x_1)\wedge P(y_1)]}{P(y_1)}\ \wedge E_2
    \qquad
    \dfrac{
      \forall E\ \dfrac{[\forall x\exists y(\neg P(x)\wedge P(y))]}{\exists y(\neg P(y_1)\wedge P(y))}
      \qquad
      \dfrac{[\neg P(y_1)\wedge P(y_2)]}{\neg P(y_1)}\ \wedge E_1
    }{\neg P(y_1)}\ \exists E
  }{\bot}\ \neg E
}{\bot}\ \exists E}{\neg(\forall x(\exists y(\neg P(x)\wedge P(y))))}\ \neg I
$$

Homework 11.1. **[Counterexamples from Sequent Calculus]** (++)

Consider the statement $\forall x(P(x) \to \neg P(f(x)))$.

1. What happens when trying to prove the validity of this formula in sequent calculus?
2. How can we derive a countermodel from the proof tree?
3. Is there a smaller countermodel?

Solution:

The proof tree gets stuck:

$$\dfrac{\dfrac{\dfrac{P(y), P(f(y)) \Rightarrow}{P(y) \Rightarrow \neg P(f(y))}\,\neg R}{\Rightarrow P(y) \to \neg P(f(y))}\,\to R}{\Rightarrow \forall x\,(P(x) \to \neg P(f(x)))}\,\forall R$$

As in the lecture, we can create a countermodel $\mathcal{A}$: Let $U_{\mathcal{A}}$ be the set of all terms over $y, f(\cdot)$, set $y^{\mathcal{A}} := y$, $f^{\mathcal{A}}(t) := f(t^{\mathcal{A}})$, and $P^{\mathcal{A}} := \{y, f(y)\}$. Then $\mathcal{A} \models P(y)$ and $\mathcal{A} \models P(f(y))$ and hence $\mathcal{A} \not\models \forall x\,(P(x) \to \neg P(f(x)))$. Note that $\mathcal{A}$ is infinite, but there are countermodels with just two elements $\{a, b\}$: Set $f(a) := b$, $f(b) := b$, $P(a)$ and $P(b)$. Then $P(a)$ and $P(f(a)) = P(b)$.

Homework 11.2. **[Natural Deduction]** (++)

Prove the following statements using natural deduction.

1. $\neg\forall x \exists y \forall z(\neg P(x,z) \land P(z,y))$
2. $\exists x(P(x) \to \forall x P(x))$

Solution:

You can ask for hints on Zulip.

Homework 11.3. [Elementary Classes] (++)

In this exercise, we assume that all structures and formulas share the same signature Σ.

We define the operator $Mod(S)$ that returns the class of all structures that model a set of formulas S. In other words, $Mod(S)$ contains all $\mathcal{A}$ such that $\mathcal{A} \models S$.

A class of models M is said to be *elementary* if there is a set of formulas S such that $M = Mod(S)$. If S is just a singleton set, i.e. there is a formula F such that $S = \{F\}$, then M is *basic elementary.*

Prove:

1. A class of models M is basic elementary if and only if there is a *finite* set of formulas S such that $M = Mod(S)$.
2. If M is basic elementary and $M = Mod(S)$ for countable S, then there is a finite subset $S' \subseteq S$ such that $M = Mod(S')$.

Solution:

For the first task, simply take $F \coloneqq \bigwedge_{G \in S} G$.

For the second task, it suffices to show that $Mod(S) = Mod(S') \iff Cn(S) = Cn(S')$. The result then follows from tutorial exercise 11.3. Here's the direction from left to right:

$$\begin{aligned} F \in Cn(S) &\iff \mathcal{M} \models F \text{ for any model } \mathcal{M} \text{ of } S \\ &\iff \mathcal{M} \models F \text{ for any } \mathcal{M} \in Mod(S) \\ &\iff \mathcal{M} \models F \text{ for any } \mathcal{M} \in Mod(S') \\ &\iff \mathcal{M} \models F \text{ for any model } \mathcal{M} \text{ of } S' \\ &\iff F \in Cn(S') \end{aligned}$$

The other direction is similar.

The logic of the world is prior to all truth and falsehood.
— Ludwig Wittgenstein[1]

[1] Yes, Ludwig strikes again – he just dropped too many great quotes.