First-order Predicate Logic
Theories

Definitions

Definition

A signature Σ is a set of predicate and function symbols.

A Σ-formula is a formula that contains only predicate and function symbols from Σ.

A Σ-structure is a structure that interprets all predicate and function symbols from Σ.

Definition

A sentence is a closed formula.

In the sequel, S is a set of sentences.

Theories

Definition

A theory is a set of sentences S such that S is closed under consequence: If $S \models F$ and F is closed, then $F \in S$.

Let $\mathcal{A}$ be a Σ-structure:
$Th(\mathcal{A})$ is the set of all sentences true in $\mathcal{A}$:
$Th(\mathcal{A}) = \{F \mid F \ \Sigma\text{-sentence and } \mathcal{A} \models F\}$

Lemma

Let $\mathcal{A}$ be a Σ-structure and F a Σ-sentence.
Then $\mathcal{A} \models F$ iff $Th(\mathcal{A}) \models F$.

Corollary

$Th(\mathcal{A})$ is a theory.

Lemma

Let $\mathcal{A}$ be a Σ-structure and F a Σ-sentence.
Then $\mathcal{A} \models F$ iff $Th(\mathcal{A}) \models F$.

Proof

"$\Rightarrow$"

Assume $\mathcal{A} \models F$

To show $Th(\mathcal{A}) \models F$, assume $\mathcal{B} \models Th(\mathcal{A})$ and show $\mathcal{B} \models F$

$\Rightarrow$ for all $G \in Th(\mathcal{A})$, $\mathcal{B} \models G$

$\Rightarrow$ $\mathcal{B} \models F$ because $F \in Th(\mathcal{A})$

"$\Leftarrow$":

Assume $Th(\mathcal{A}) \models F$

$\Rightarrow$ for all $\mathcal{B}$, if $\mathcal{B} \models Th(\mathcal{A})$ then $\mathcal{B} \models F$

$\Rightarrow$ $\mathcal{A} \models F$ because $\mathcal{A} \models Th(\mathcal{A})$

Example

Notation: $(\mathbb{Z}, +, \leq)$ denotes the structure with universe $\mathbb{Z}$ and the standard interpretations for the symbols $+$ and $\leq$.
The same notation is used for other standard structures where the interpretation of a symbol is clear from the symbol.

Example (Linear integer arithmetic)

$Th(\mathbb{Z}, +, \leq)$ is the set of all sentences over the signature $\{+, \leq\}$ that are true in the structure $(\mathbb{Z}, +, \leq)$.

Famous numerical theories

$Th(\mathbb{R}, +, \leq)$ is called linear real arithmetic.
It is decidable.

$Th(\mathbb{R}, +, *, \leq)$ is called real arithmetic.
It is decidable.

$Th(\mathbb{Z}, +, \leq)$ is called linear integer arithmetic or Presburger arithmetic.
It is decidable.

$Th(\mathbb{Z}, +, *, \leq)$ is called integer arithmetic.
It is not even semidecidable (= r.e.).

Decidability via special algorithms.

Consequences

Definition

Let S be a set of Σ-sentences.

$Cn(S)$ is the set of consequences of S:
$Cn(S) = \{F \mid F\ \Sigma\text{-sentence and } S \models F\}$

Examples

$Cn(\emptyset)$ is the set of valid sentences.
$Cn(\{\forall x \forall y \forall z\ (x * y) * z = x * (y * z)\})$ is the set of sentences that are true in all semigroups.

Lemma

If S is a set of Σ-sentences, $Cn(S)$ is a theory.

Proof Assume F is closed and $Cn(S) \models F$. Show $F \in Cn(S)$, i.e. $S \models F$. Assume $\mathcal{A} \models S$. Thus $\mathcal{A} \models Cn(S)$ (*) and hence $\mathcal{A} \models F$, i.e. $S \models F$. (*): Assume $G \in Cn(S)$, i.e. $S \models G$. With $\mathcal{A} \models S$ the desired $\mathcal{A} \models G$ follows.

Axioms

Definition

Let S be a set of Σ-sentences.

A theory T is axiomatized by S if $T = Cn(S)$

A theory T is axiomatizable if there is some decidable or recursively enumerable S that axiomatizes T.

A theory T is finitely axiomatizable
if there is some finite S that axiomatizes T.

Completeness and elementary equivalence

Definition
A theory T is complete if for every sentence F, $T \models F$ or $T \models \neg F$.

Fact
$Th(A)$ is complete.

Example
$Cn(\{\forall x \forall y \forall z\ (x * y) * z = x * (y * z)\})$ is incomplete:
neither $\forall x \forall y\ x * y = y * x$ nor its negation are present.

Definition
Two structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if
$Th(\mathcal{A}) = Th(\mathcal{B})$.

Theorem
A theory T is complete iff all its models are elementarily equivalent.

Theorem

A theory T is complete iff all its models are elementarily equivalent.

Proof If T is unsatisfiable, then T is complete (because $T \models F$ for all F) and all models are elementarily equivalent.
Now assume T has a model $\mathcal{M}$.
"$\Rightarrow$"
Assume T is complete. Let $F \in Th(\mathcal{M})$.
We cannot have $T \models \neg F$ because $\mathcal{M} \models T$ would imply $\mathcal{M} \models \neg F$ but $\mathcal{M} \models F$ because $F \in Th(\mathcal{M})$. Thus $T \models F$ by completeness.
Therefore every formula that is true in some model of T is true in all models of T.
"$\Leftarrow$"
Assume all models of T are elem.eq. Let F be closed.
Either $\mathcal{M} \models F$ or $\mathcal{M} \models \neg F$. By elem.eq. $T \models F$ or $T \models \neg F$.
Why? Assume $\mathcal{M} \models F$ (similar for $\mathcal{M} \models \neg F$).
To show $T \models F$, assume $\mathcal{A} \models T$ and show $\mathcal{A} \models F$.
$\Rightarrow Th(A) = Th(\mathcal{M})$ by elem.eq.
$\Rightarrow$ for all closed F, $\mathcal{A} \models F$ iff $\mathcal{M} \models F$
$\Rightarrow \mathcal{A} \models F$ because $\mathcal{M} \models F$