First-Order Logic

The Classical Decision Problem

Validity/satisfiability of arbitrary first-order formulas is undecidable.

What about subclasses of formulas?

Examples

$\forall x \exists y \, (P(x) \to P(y))$

Satisfiable? Resolution?

$\exists x \forall y \, (P(x) \to P(y))$

Satisfiable? Resolution?

The $\exists^*\forall^*$ class

Definition

The $\exists^*\forall^*$ class is the class of closed formulas of the form

$$\exists x_1 \ldots \exists x_m \forall y_1 \ldots \forall y_n \; F$$

where F is quantifier-free and contains no function symbols of arity > 0.

This is also called the Bernays-Schönfinkel class.

Corollary

Unsatisfiability is decidable for formulas in the $\exists^*\forall^*$ *class.*

What if a formula is not in the $\exists^*\forall^*$ class?
Try to transform it into the $\exists^*\forall^*$ class!

Example

$\forall y \exists x \, (P(x) \wedge Q(y))$

Heuristic transformation procedure:

1. Put formula into NNF
2. Push all quantifiers into the formula as far as possible ("miniscoping")
3. Pull out $\exists$ first and $\forall$ afterwards

Miniscoping

Perform the following transformations bottom-up,
as long as possible:

- $(\exists x\, F) \;\equiv\; F$ if x does not occur free in F
- $\exists x\,(F \vee G) \;\equiv\; (\exists x\, F) \vee (\exists x\, G)$
- $\exists x\,(F \wedge G) \;\equiv\; (\exists x\, F) \wedge G$ if x is not free in G
- $\exists x\, F$ where F is a conjunction,
  x occurs free in every conjunct,
  and the DNF of F is of the form $F_1 \vee \cdots \vee F_n$, $n \geq 2$:
  $\exists x\, F \;\equiv\; \exists x\,(F_1 \vee \cdots \vee F_n)$

Together with the dual transformations for $\forall$

Example

$\exists x\,(P(x) \wedge \exists y\,(Q(y) \vee R(x)))$

Warning: Complexity!

The monadic class

Definition

A formula is monadic if it contains only unary (monadic) predicate symbols and no function symbol of arity > 0.

Examples

All men are mortal. Sokrates is a man. Sokrates is mortal.

The monadic class is decidable

Theorem

Satisfiability of monadic formulas is decidable.

Proof Put into NNF. Perform miniscoping.
The result has no nested quantifiers (Exercise!).
First pull out all $\exists$, then all $\forall$.
Existentially quantify free variables.
The result is in the $\exists^*\forall^*$ class.

Corollary

Validity of monadic formulas is decidable.

The finite model property

Definition
A formula F has the finite model property (for satisfiability) if F has a model iff F has a finite model.

Theorem
If a formula has the finite model property, satisfiability is decidable.

Theorem
Monadic formulas have the finite model property.

The finite model property

Theorem

Monadic formulas have the finite model property.

Proof A monadic formula F
with k different monadic predicate symbols $P_1, \dots, P_k$
has a model of size $\leq 2^k$.
Given a model $\mathcal{A}$ of F, define $\sim$ such that $|U_{\mathcal{A}/\sim}| \leq 2^k$:

$u \sim v$ iff for all i, $P_i^{\mathcal{A}}(u) = P_i^{\mathcal{A}}(v)$

Why $|U_{\mathcal{A}/\sim}| \leq 2^k$?
Every class $[u]_\sim$ can be viewed as a bit-vector of length k:
$(P_1^{\mathcal{A}}(u), \dots, P_k^{\mathcal{A}}(u))$

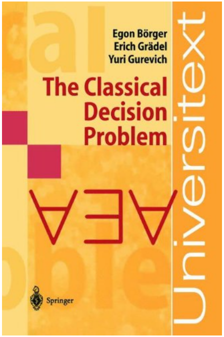
Obvious: $\sim$ is an equivalence.

$\sim$ is a congruence: if $u \sim v$ then $P_i^{\mathcal{A}}(u) = P_i^{\mathcal{A}}(u)$ for all i

Classification by quantifier prefix of prenex form

There is a complete classification of decidable and undecidable classes of formulas based on

- the form of the quantifier prefix of the prenex form
- the arity of the predicate and function symbols allowed
- whether "$=$" is allowed or not.



A complete classification

Only formulas without function symbols of arity > 0,
no restrictions on predicate symbols.

Satisfiability is decidable:

$\exists^*\forall^*$ (Bernays, Schönfinkel 1928, Ramsey 1930)

$\exists^*\forall\exists^*$ (Ackermann 1928)

$\exists^*\forall^2\exists^*$ (Gödel 1932)

Satsifiability is undecidable:

$\forall^3\exists$ (Surányi 1959)

$\forall\exists\forall$ (Kahr, Moore, Wang 1962)

Why complete?

Famous mistake by Gödel: $\exists^*\forall^2\exists^*$ with "$=$" is undecidable
(Goldfarb 1984)