

LOGIC EXERCISES

TECHNICAL UNIVERSITY OF MUNICH
CHAIR FOR LOGIC AND VERIFICATION

PROF. TOBIAS NIPKOW
KEVIN KAPPELMANN

SS 2022

EXERCISE SHEET 7

10.06.2022

Exercise 7.1. [(In)finite Models]

Consider predicate logic with equality. We use infix notation for equality and abbreviate $\neg(s = t)$ by $s \neq t$. Moreover, we call a structure finite if its universe is finite.

1. Specify a finite model for the formula $\forall x (c \neq f(x) \wedge x \neq f(x))$.
2. Specify a model for the formula $\forall x \forall y (c \neq f(x) \wedge (f(x) = f(y) \longrightarrow x = y))$.
3. Show that the second formula has no finite model.

Solution:

1. $U^{\mathcal{A}} = \{0, 1, 2\} \subset \mathbb{N}$, $c^{\mathcal{A}} = 0$, $f^{\mathcal{A}}(0) = 1$, and $f^{\mathcal{A}}(n + 1) = 2 - n$
2. $U^{\mathcal{A}} = \mathbb{N}$ and $c^{\mathcal{A}} = 0$ and $f^{\mathcal{A}}(n) = n + 1$
3. Assume \mathcal{M} is a finite model of the formula. By the second conjunct, $f^{\mathcal{M}}$ is injective so $|U_{\mathcal{M}}| \leq |f(U_{\mathcal{M}})|$. Further, $f(U_{\mathcal{M}}) \subseteq U_{\mathcal{M}}$ so $|f(U_{\mathcal{M}})| = |U_{\mathcal{M}}|$ and hence $f(U_{\mathcal{M}}) = U_{\mathcal{M}}$ (using our finiteness assumption). Thus there is $d \in U_{\mathcal{M}}$ such that $f(d) = c^{\mathcal{M}}$, contradicting the first conjunct.

Exercise 7.2. [Herbrand Structures]

Consider the formula

$$F = \forall x \forall y (P(f(x), g(y)) \wedge \neg P(g(x), f(y)))$$

1. Specify a Herbrand model for F .
2. Specify a Herbrand structure suitable for F that is not a model of F .

Solution:

We define $U_{\mathcal{A}} = T(F)$, i.e., the Herbrand universe for F . We invent a constant $a \in T(F)$. We define $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ to be the Herbrand interpretations.

1. $P^{\mathcal{A}} = \{(f(t_1), g(t_2)) \mid t_1, t_2 \in T(F)\}$.
2. $P^{\mathcal{A}} = \{(g(t_1), f(t_2)) \mid t_1, t_2 \in T(F)\}$.

Exercise 7.3. [Ground Resolution]

Use ground (Gilmore) resolution to prove that the following formula is valid:

$$(\forall xP(x, f(x))) \longrightarrow \exists yP(c, y)$$

Solution:

First put the formula into Skolem form:

$$\begin{aligned} & \neg((\forall xP(x, f(x))) \longrightarrow \exists yP(c, y)) \\ & (\forall xP(x, f(x))) \wedge \neg\exists yP(c, y) && \text{(push negation)} \\ & (\forall xP(x, f(x))) \wedge \forall y\neg P(c, y) && \text{(push negation)} \\ & \forall x\forall y(P(x, f(x)) \wedge \neg P(c, y)) && \text{(Skolem-Form)} \end{aligned}$$

Now enumerate the Herbrand expansion:

$$CE(F) = \{P(c, f(c)), \neg P(c, f(c)), \dots\}$$

With resolution, we immediately get \square from the first two items in the enumeration.

Exercise 7.4. [Uncountable “Natural Numbers”]

We consider the following axioms in an attempt to model the natural numbers in first-order logic with equality:

1. $F_1 = \forall x \forall y (f(x) = f(y) \rightarrow x = y)$
2. $F_2 = \forall x (f(x) \neq 0)$
3. $F_3 = \forall x (x = 0 \vee \exists y (x = f(y)))$

Give a model with an *uncountable* universe for:

1. $\{F_1, F_2\}$
2. $\{F_1, F_2, F_3\}$

Remember: A set S is uncountable if there is no injection from S to \mathbb{N} .

Solution:

1. $U_{\mathcal{A}} = \mathbb{R}_0^+$, $0^{\mathcal{A}} = 0$, and $f^{\mathcal{A}}(x) = x + 1$
 $f^{\mathcal{A}}$ is clearly injective and there is no x such that $f^{\mathcal{A}}(x) = 0$, because $-1 \notin U_{\mathcal{A}}$.
2. We take $U_{\mathcal{A}}$ to be the union of the positive real numbers and the non-positive whole numbers, i.e., $U_{\mathcal{A}} = \mathbb{R}_{>0} \cup \mathbb{Z}_{\leq 0}$.

Let the symbols be interpreted as follows:

$$0^{\mathcal{A}} = 0$$

$$f^{\mathcal{A}}(x) = \begin{cases} 2x & \text{if } x > 0 \\ x - 1 & \text{if } x \leq 0 \end{cases}$$

- (a) $f^{\mathcal{A}}$ is defined as two disjoint domains that have disjoint ranges and f is injective on both domains; hence the entire function is injective.
- (b) 0 is not in the range of $f^{\mathcal{A}}$: For $x > 0$, $f^{\mathcal{A}}(x) > 0$ and for $x \leq 0$, $f^{\mathcal{A}}(x) \leq -1$.
- (c) To show: $x \neq 0 \rightarrow \exists y (x = f(y))$.
 If $x < 0$, then $x \leq -1$, hence $x = f^{\mathcal{A}}(x + 1)$.
 Otherwise, $x = f^{\mathcal{A}}(\frac{x}{2})$.

Homework 7.1. [Model Sizes] (++)

1. Specify a satisfiable formula F (one with and one without equality) such that for all models \mathcal{A} of F , we have $|U_{\mathcal{A}}| \geq 4$.
2. Can you also specify a satisfiable formula F such that for all models \mathcal{A} of F , we have $|U_{\mathcal{A}}| \leq 4$? Again, consider both predicate logic with and without equality.
3. Specify a satisfiable formula F with equality such that for all finite models \mathcal{A} of F , we have $|U_{\mathcal{A}}| \in 2\mathbb{N}_{>0}$.

Solution:

1. $\exists x_1, x_2, x_3, x_4. x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_1 \neq x_4 \wedge x_2 \neq x_3 \wedge x_2 \neq x_4 \wedge x_3 \neq x_4$

Without equality, one can use two predicates B_1 and B_0 representing bits and 4 constants a_0, a_1, a_2, a_3 and for each a_i , encode i in binary using B_1 and B_0 , i.e. $\neg B_1(a_0) \wedge \neg B_1(a_0) \wedge \neg B_1(a_1) \wedge B_1(a_1) \wedge B_1(a_2) \wedge \neg B_1(a_2) \dots$

2. $\forall x \forall y. x = y$

Without equality, no limit on the model size can be given: assume there is a finite model \mathcal{M} . Take an arbitrary element e of \mathcal{M} . We can add a new element e_1 that behaves exactly the same as e to \mathcal{M} without changing the set of formula satisfied by \mathcal{M} (proof by structural induction on formulas). We continue this process iteratively to obtain a model of arbitrary, countable size. Indeed, we could even add an infinite number of copies of e and obtain models of arbitrary size.

3. $\forall x \neg P(x, x) \wedge \forall x \exists y P(x, y) \wedge \forall x \forall y \forall z (P(x, y) \wedge P(y, z) \rightarrow (x = z))$

Homework 7.2. [Herbrand Structures] (+)

Consider the formula

$$F = \forall x (P(f(x)) \leftrightarrow \neg P(x))$$

1. Specify a Herbrand model for F .
2. Specify a Herbrand structure suitable for F that is not a model of F .

Homework 7.3. [Preconditions Are Here To Stay] (+)

Recall the fundamental theorem from the lecture: “Let F be a closed formula in Skolem form. Then F is satisfiable iff it has a Herbrand model”.

Explain: what goes wrong if the precondition is violated, that is when F is not closed or not in Skolem form. Describe both cases.

Solution:

$\exists x. (P(x) \wedge \neg P(a))$; there is no Herbrand model because there is only the constant a and no functionals but we need at least two elements for the formula to be satisfiable. The same problem arises for $P(x) \wedge \neg P(a)$.

Homework 7.4. [Ground resolution] (++)

Execute ground resolution to show that the following formula is unsatisfiable:

$$\forall x \forall y ((P(x) \wedge \neg Q(y, y)) \rightarrow Q(x, y)) \wedge \neg \exists x (P(x) \wedge \exists y (Q(y, y) \wedge Q(x, y))) \wedge \exists y (P(y))$$

Solution:

Algorithmic.

Homework 7.5. [Proof of the Fundamental Theorem] (++)

Recall the fundamental theorem: Let F be a closed formula in Skolem form. Then F is satisfiable iff it has a Herbrand model. Give the omitted proof for the base case (slide 6, $\mathcal{A}(G) = \mathcal{T}(G)$).

Solution:

Let \mathcal{A} be an arbitrary model of F . We define a Herbrand structure \mathcal{T} as follows (according to the lecture):

$$\begin{aligned} U_{\mathcal{T}} &= T(F) & f^{\mathcal{T}}(t_1, \dots, t_n) &= f(t_1, \dots, t_n) \\ (t_1, \dots, t_n) \in P^{\mathcal{T}} &\text{ iff } (\mathcal{A}(t_1), \dots, \mathcal{A}(t_n)) \in P^{\mathcal{A}} \end{aligned}$$

Additionally, if F contains no constant: $a^{\mathcal{A}} = u$ for some arbitrary $u \in U_{\mathcal{A}}$.

We now prove the omitted case for the following stronger proposition: For every closed formula G in Skolem form such that all function and predicate symbols in G occur in F , if $\mathcal{A} \models G$ then $\mathcal{T} \models G$. The proof proceeds by induction on the number n of universal quantifiers in G .

- Base case: $n = 0$. G has no quantifiers (because it is in Skolem form).

Claim: $\mathcal{A}(G) = \mathcal{T}(G)$.

Proof by induction on the structure of G .

- Base case: $G = P(t_1, \dots, t_k)$

We know that $\mathcal{A}(\mathcal{T}(t)) = \mathcal{A}(t)$, because $\mathcal{T}(t) = t$.

$$\begin{aligned} \mathcal{T} \models P(t_1, \dots, t_k) &\text{ iff } (\mathcal{T}(t_1), \dots, \mathcal{T}(t_k)) \in P^{\mathcal{T}} \\ &\text{ iff } (\mathcal{A}(\mathcal{T}(t_1)), \dots, \mathcal{A}(\mathcal{T}(t_k))) \in P^{\mathcal{A}} \\ &\text{ iff } (\mathcal{A}(t_1), \dots, \mathcal{A}(t_k)) \in P^{\mathcal{A}} \\ &\text{ iff } \mathcal{A} \models P(t_1, \dots, t_k) \end{aligned}$$

- Induction step: $G = H_1 \wedge H_2$

Induction hypotheses: $\mathcal{A}(H_i) = \mathcal{T}(H_i)$ for $i \in \{1, 2\}$

$$\begin{aligned} \mathcal{A}(H_1 \wedge H_2) &\text{ iff } \mathcal{A}(H_1) \text{ or } \mathcal{A}(H_2) \\ &\text{ iff } \mathcal{T}(H_1) \text{ or } \mathcal{T}(H_2) \\ &\text{ iff } \mathcal{T}(H_1 \wedge H_2) \end{aligned}$$

- Other induction steps are similar.

Logic takes care of itself; all we have to do is to look and see how it does it.
— Ludwig Wittgenstein