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Equational Logic
Summer Term 2012
Exercise Sheet 11
July 6

Homework is due on July 13th, before the tutorial.

Exercise 1 (H) (*Church-Encodings: Trees*)

a) Encode a datatype of binary trees in lambda calculus. Specify the operations `tip` and `node` that construct trees, as well as `isTip`, `left`, and `right`. Show that the following holds:

$$\begin{aligned} \texttt{isTip}\;\texttt{tip} &\to^* \texttt{true} \\ \texttt{isTip}\;(\texttt{node}\;x\;y) &\to^* \texttt{false} \\ \texttt{left}\;(\texttt{node}\;x\;y) &\to^* x \\ \texttt{right}\;(\texttt{node}\;x\;y) &\to^* y \end{aligned}$$

Exercise 2 (H) (*Parallel β-Reduction*)

In the lecture, we defined parallel β-reduction $>$ inductively as follows:

$$\begin{array}{rcrcl} & & s & > & s \\ s > s' & \Longrightarrow & \lambda x.s & > & \lambda x.s' \\ s > s' \;\wedge\; t > t' & \Longrightarrow & (s\;t) & > & (s'\;t') \\ s > s' \;\wedge\; t > t' & \Longrightarrow & (\lambda x.s)\;t & > & s'\;[t'/x] \end{array}$$

We also showed $> \subseteq \overset{*}{\longrightarrow}_\beta$. In this exercise, you have to show: $\longrightarrow_\beta \subseteq >$

Hint: Use induction over the inductive definition of $\longrightarrow_\beta$ (Def. 1.2.2).

Exercise 3 (T) (*Lists*)

Specify λ-Terms for `nil`, `cons`, `hd`, `tl` and `null`, that enocde lists in the λ-calculus. Show that your terms satisfy the following:

$$\begin{array}{lll lll} \texttt{null}\;\texttt{nil} & \longrightarrow^* & \texttt{true} & \texttt{hd}\;(\texttt{cons}\;x\;l) & \longrightarrow^* & x \\ \texttt{null}\;(\texttt{cons}\;x\;l) & \longrightarrow^* & \texttt{false} & \texttt{tl}\;(\texttt{cons}\;x\;l) & \longrightarrow^* & l \end{array}$$

Hint: Use pairs.

Exercise 4 (T) (*Confluence of β-reduction*)

In the lecture we have shown the confluence of $\longrightarrow_\beta$ using the diamond property of parallel β-reduction (cf. Exercise 2). In this exercise, we develop an alternative proof.

We define the operation $*$ on λ-terms inductively over the structure of terms:

$$\begin{array}{rcll} x^* & = & x & \\ (\lambda x.\, t)^* & = & \lambda x.\, t^* & \\ (t_1 t_2)^* & = & t_1^* t_2^* & \text{if } t_1 t_2 \text{ is not } \beta\text{-reducible.} \\ ((\lambda x.\, t_1) t_2)^* & = & t_1^*[t_2^*/x] & \end{array}$$

a) Show that we have for two arbitrary λ-terms s and t: $s > t \quad \Longrightarrow \quad t > s^*$

b) Show that $\longrightarrow_\beta$ is confluent.