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Equational Logic
Summer Term 2015
Exercise Sheet 1

Every sheet contains *exercises* and *homework* assignments. We strongly recommend you prepare for the exercise sessions by reading the exercises on the sheet and make yourself familiar with the concepts. Homework assignments are due the following week after the sheet was published, to be handed in before the exercise session. You have to do the homework assignments yourself. Team work is not allowed!

Exercise 1 (Warm-Up)

Which of the following closure operators commute? Prove or refute!

a) $\longleftrightarrow^{+} = \xrightarrow{+} \cup (\xrightarrow{+})^{-1}$

b) $\longleftarrow^{+} = (\xrightarrow{+})^{-1}$

Exercise 2 (Bounded Relations)

A relation $\longrightarrow$ over the set A is called *bounded*, if for each element x, the lengths of all paths from x are bounded. Formally:

$$\forall x \in A.\ \exists n.\ \nexists y \in A.\ x \xrightarrow{n} y$$

Prove or refute:

a) Each terminating relation is bounded.

b) A finitely branching relation is terminating if and only if it is bounded. (Hint: Well-founded induction)

c) Now we call a relation *globally bounded*, if there is a bound that is valid for all elements. Formally:

$$\exists n.\ \forall x \in A.\ \nexists y \in A.\ x \xrightarrow{n} y$$

Prove or refute: Any finitely branching and terminating relation is globally bounded.

Exercise 3 (Partial Ordering)

Prove or refute:

a) $\xrightarrow{+}$ is a strict partial order if and only if $\longrightarrow$ is acyclic.

b) $\xrightarrow{*}$ is a partial order if and only if $\longrightarrow$ is acyclic.

Notes: A relation $R \subseteq X \times X$ is called *strict partial order* if it is irreflexive ($\forall x \in X.\ \neg(x\ R\ x)$), transitive ($\forall x, y, z \in X.\ x\ R\ y \wedge y\ R\ z \implies x\ R\ z$), and asymmetric ($\forall x, y \in X.\ x\ R\ y \implies \neg(y\ R\ x)$).

A relation $R \subseteq X \times X$ is called *partial order* if it is reflexive ($\forall x \in X.\ x\ R\ x$), transitive and antisymmetric ($\forall x, y \in X.\ x\ R\ y \wedge y\ R\ x \implies x = y$).

A relation $\longrightarrow \subseteq X \times X$ is called *acyclic* if there is no element a, s.t. $a \xrightarrow{+} a$.

Exercise 4 (Example)

Let $(M, \longrightarrow)$ be a reduction system with $M = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, C_1, C_2, C_3, C_4, D, E\}$ and $\longrightarrow$ defined as follows:

- $A_1 \longrightarrow B_1$, $A_1 \longrightarrow B_2$, $A_2 \longrightarrow B_1$, $A_2 \longrightarrow B_2$, $A_3 \longrightarrow B_3$, $A_4 \longrightarrow B_3$,
- $B_1 \longrightarrow C_1$, $B_2 \longrightarrow C_2$, $B_2 \longrightarrow C_3$, $B_3 \longrightarrow C_1$, $B_3 \longrightarrow C_2$, $B_3 \longrightarrow C_3$, $B_3 \longrightarrow C_4$,
- $C_3 \longrightarrow E$, $C_4 \longrightarrow E$, and $D \longrightarrow C_4$.

Which of the following properties are satisfied by $\longrightarrow$? Give a justification.
terminating, globally bounded, asymmetric, antisymmetric, reflexive, irreflexive, transitive

Homework 5 (Primes)

Let $(\mathbb{N}_{>0}, \longrightarrow)$ be the reduction system on positive natural numbers, where

$$\longrightarrow = \{(n, m) \mid 11n = 2m \vee 5n = 13m\}$$

a) Does this system terminate? Justify your answer.

b) Determine the set of all irreducible elements.

c) What is the normal form of 1210? Show: $10 \longleftrightarrow 26$ and $10 \overset{*}{\longleftrightarrow} 143$.

Homework 6 (Equivalence Relation)

A relation $R \subseteq X \times X$ is called an *equivalence relation*, if:

- R is reflexive, i.e. $\forall x \in X.\ x\ R\ x$
- R is transitive, i.e. $\forall x, y, z \in X.\ x\ R\ y \wedge y\ R\ z \implies x\ R\ z$
- R is symmetric, i.e. $\forall x, y \in X.\ x\ R\ y \implies y\ R\ x$

Let $\longrightarrow$ be a relation. Show: $\overset{*}{\longleftrightarrow}$ is the smallest equivalence relation that contains $\longrightarrow$.

Homework 7 (Confluence And Normal Form)

a) Show that a reduction system $(A, \longrightarrow)$ is confluent and normalizing, if and only if every element has a unique normal form.

b) Give an example of a confluent and normalizing reduction system that does not terminate.