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Equational Logic
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Solutions to Exercise Sheet 7

Exercise 34 (Termination)

Let $\Sigma = \{a, b, c, d\}$, with unary function symbols a, b und c and a constant symbol d. Show that the term rewriting system with the following rules terminates:

$$\begin{aligned} b(a(x)) &\longrightarrow a(b^2(c(x))) \\ c(a(x)) &\longrightarrow a(b(c^2(x))) \\ c(b(x)) &\longrightarrow b(c(x)) \end{aligned}$$

Hint: Consider how the number of occurences of as changes in each step. Then regard the sequences of function symbols in between the as as strings.

Solution

All functions in Σ are unary, and terms in $T(\Sigma, V)$ can be identified by words from Σ^*, if we drop variables and the constant symbol d. This simplifies the notation in the following proof.

Let $v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_i \longrightarrow \ldots$ be a reduction sequence. The letter a occurs equally often in each v_i, and the v_i have the form

$$w_{i0}aw_{i1}a \cdots aw_{i,n-1}aw_{in}, \qquad w_{ij} \in (\Sigma - \{a\}) \tag{*}$$

In a reduction, the length of w_{i0} is bounded by the length of w_{00}. Thus, the first and second rule can only be applied infinitely often on the first a. Hence, also the length of w_{i1} is bounded, and so is the length of $w_{i2}, \ldots, w_{in}$.

We define an ordering on Σ by $c > b > a$. This yields the lexicographic order $>_{\text{lex}}$ on Σ^*. This order does not necessarily terminate, (e.g. $b >_{\text{lex}} ab >_{\text{lex}} a^2b >_{\text{lex}} \cdots$), but it terminates for bounded word length.

The order $>_{\text{lex,lex}}$ is the order on n-tuples of words from Σ^*, that is induced by $>_{\text{lex}}$. We can apply this order on words of the form (*) by identifying v_i with $(w_{i0}, \ldots, w_{in})$. This order is terminating, as the lengths of the w_{ij} are bounded.

It remains to show that the rules are compatible with $>_{\text{lex,lex}}$. Let j be the occurence of a in (*), or the subword, on which the rule is applied.

$b(a(x)) \longrightarrow a(b^2(c(x)))$: $(\ldots, w_{ij}b, w_{i,j+1}, \ldots) >_{\text{lex,lex}} (\ldots, w_{ij}, b^2cw_{i,j+1}, \ldots)$, as the word at position j gets shorter, and thus smaller w.r.t. $>_{\text{lex}}$.

$c(a(x)) \longrightarrow a(b(c^2(x)))$: $(\ldots, w_{ij}c, w_{i,j+1}, \ldots) >_{\text{lex,lex}} (\ldots, w_{ij}, bc^2w_{i,j+1}, \ldots)$, analogously.

$c(b(x)) \longrightarrow b(c(x))$**:** $(\ldots, w_{ij}, \ldots) >_{\text{lex,lex}} (\ldots, w_{i+1,j}, \ldots)$, as at position j, an occurence of cb is replaced by bc, and thus the word gets smaller w.r.t. $>_{\text{lex}}$.

Exercise 35 (Hilbert's 10th Problem (Exercise 5.8 of TRaAT))

Show that undecidability of Hilbert's 10th Problem implies that the following problem (TRaT Exercise 5.8) is undecidable:

Instance: Two polynomials $P, Q \in \mathbb{N}[X_1, \ldots, X_n]$ in n indeterminates with non-negative integer coefficients, and a (decidable) subset A of $\mathbb{N}$.

Question: Does $P >_A Q$ hold, i.e. is the value of P greater than the value of Q for all valuations with elements in A.

Show that this implies that there exists a polynomial interpretation $\mathcal{A}$ for which it is in general undecidable whether two terms l, r satisfy $l >_{\mathcal{A}} r$.

Solution

Theorem TRaT Exercise 5.8 is undecidable.

Proof by contradiction. We assume our problem is decidable, i.e. for each polynomials P, and Q, and for each decidable set A, we can decided $P <_A Q$.

We now show how to reduce instances of Hilbert's 10th problem to our problem: Given a polynom $P \in \mathbb{Z}[X_1, \ldots, X_n]$ (now in the integers), decide $\exists \mathbf{x} \in \mathbb{N}^n.\ P(\mathbf{x}) = 0$. We construct two polynomials $R, Q \in \mathbb{N}[X_1, \ldots, X_n]$ with:

$$P^2(\mathbf{x}) = R(\mathbf{x}) - Q(\mathbf{x})$$

This is simply done by chosing all positive coefficients of P^2 for R and all negative ones for Q. Now we decide $R >_{\mathbb{N}} Q$, which is equivalent to $P^2 > 0$, and hence to $\forall \mathbf{x} \in \mathbb{N}^n.\ P(\mathbf{x}) > 0$. Hence, we would decide Hilbert's 10th problem. **Contradiction.**

Theorem $l >_{\mathcal{A}} r$ is generally not decidable for arbitrary $T(\Sigma, V)$ and $\mathcal{A}$.

Proof.

As signature we choose $\Sigma = \{+_2, *_2, 1_0\}$, as interpretation we choose $P_+(x, y) = x + y$, $P_* = x \cdot y$, and $P_1 = 1$, as polynom interpretations in $\mathcal{A}$. Each TRaT Exercise 5.8 instance, Q, R, A can now be translated into a polynomial order problem:

$$Q <_A R \Longleftrightarrow_{\text{Def}} P_{Q_{\mathcal{A}}} < P_{R_{\mathcal{A}}} \Longleftrightarrow_{5.3.8} Q <_{\mathcal{A}} R$$

We write $Q_{\mathcal{A}}$ for the term with $P_{Q_{\mathcal{A}}}(\mathbf{x}) = Q(\mathbf{x})$, i.e. the term representing the polynomial Q. This proves the first equation, the second equation is Lemma 5.3.8 in TRaAT. **Contradiction.**

Homework 36 (Reduction Ordering)

Recall that a reduction ordering is a well-founded ordering on terms that is compatible with context and closed under substitutions. Now consider the subterm ordering $>_{ST}$, defined so that $s >_{ST} t$ iff t is a proper subterm of s.

a) Show that $>_{ST}$ is no reduction ordering.

b) Show that a term-rewriting system R with $R \subseteq >_{ST}$ always terminates. Here, $R \subseteq >_{ST}$ means that $l >_{ST} r$ for every rewrite rule $(l \longrightarrow r) \in R$.

Homework 37 (Polynomial Interpretation)

Use the polynomial interpretation $\mathcal{A}$ with $A = \mathbb{N} - \{0, 1, 2\}$ and $P_f(X, Y) = X^2 + XY$to show that the following term rewriting system terminates:

$$\{\ f(f(x, y), z) \longrightarrow f(x, f(y, z)),\ f(x, f(y, z)) \longrightarrow f(y, y)\ \}$$

Homework 38 (Interpretation)

Prove termination of the following term rewriting system using the interpretation method:

$$\{f(f(x)) \longrightarrow f(g(f(x)))\}$$