Technische Universität München
Institut für Informatik
Prof. Tobias Nipkow, Ph.D.
Johannes Hölzl and Lars Hupel

Equational Logic
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Solutions to Exercise Sheet 8

Exercise 39 (Critical Pairs)

Specify all critical pairs of the following term rewriting systems:

a) $f(g(f(x))) \longrightarrow x,\ f(g(x)) \longrightarrow g(f(x))$

b) $g(f(x)) \longrightarrow f(g^2(h(x))),\ h(f(x)) \longrightarrow f(g(h^2(x))),\ h(g(x)) \longrightarrow g(h(x))$

Which systems are locally confluent, which are convergent (i.e., terminating and confluent)?

Solution

a) Three critical pairs:

$$\begin{tikzcd}
& x \\
\overline{\underline{f(g(}f}(x))) \arrow[ur] \arrow[dr] & \\
& g(f(f(x)))
\end{tikzcd}
\qquad
\begin{tikzcd}
& g(x) \\
\overline{f(g(\underline{f}}\underline{(g}(x)))) \arrow[ur] \arrow[dr] & \\
& f(g(g(f(x))))
\end{tikzcd}$$

$$\begin{tikzcd}
& g(f(x)) \\
\overline{f(g(\underline{f}}\underline{(g(f}(x))))) \arrow[ur] \arrow[dr] & \\
& f(g(x)) \arrow[u]
\end{tikzcd}$$

Not locally confluent, not convergent.

b) One critical pair:

$$\begin{tikzcd}
& g(h(f(x))) \arrow[r] & g(f(g(h^2(x)))) \arrow[r] & f(g^2(h(g(h^2(x))))) \arrow[d, "*"] \\
\overline{h(\underline{g}}\underline{(f}(x))) \arrow[ur] \arrow[dr] & & & f(g^3(h^3(x))) \\
& h(f(g^2(h(x)))) \arrow[r] & f(g(h^2(g^2(h(x))))) \arrow[ur, "*"] &
\end{tikzcd}$$

Locally confluent, terminating, hence convergent.

Exercise 40 (Confluence)

Determine terms r_1 and r_2 such that $\{f(g(x)) \longrightarrow r_1,\ g(h(x)) \longrightarrow r_2\}$ is confluent.

Solution

- A trivial solution is $r_1 = f(g(x))$ and $r_2 = g(h(x))$. This is obviously confluent, but does not terminate.
- An alternative solution is described in the following:

  R has a critical pair:

$$\begin{array}{ccccc} & & r_1[h(x)] & & \\ & \nearrow & & \searrow & \\ \overline{f(}\underline{g(h}(x))) & & & & ? \quad t \\ & \searrow & & \nearrow & \\ & & f(r_2[x]) & & \end{array}$$

  Thus, r_1 and r_2 have to be choosen such that:

  $r_1[h(x)] \overset{*}{\rightarrow} t \overset{*}{\leftarrow} f(r_2[x])$

  With $r_1 = g(x)$ and $r_2 = g(x)$ we get:

$$\begin{array}{ccccc} & & g(h(x)) & & \\ & \nearrow & & \searrow & \\ \overline{f(}\underline{g(h}(x))) & & & & g(x) \\ & \searrow & & \nearrow & \\ & & f(g(x)) & & \end{array}$$

  Hence, R is confluent.

Exercise 41 (Newman's Lemma)

Give an indirect proof of Newman's Lemma, by showing that $\longrightarrow$ has an infinite reduction sequence, if $\longrightarrow$ is locally confluent but not confluent.

Hint: Show that every element with two distinct normal forms has a direct successor with two distinct normal forms.

Solution

Proof. Let $\longrightarrow$ be locally confluent but not confluent. We assume that $\longrightarrow$ terminates and derive a contradiction.

$$\begin{array}{lll} \text{not confluent} & \Rightarrow & \exists x \text{ with two distinct normal forms } \mathrm{NF}_1 \text{ and } \mathrm{NF}_2 \\ & \Rightarrow & x \text{ has at least two direct successors: } a \longleftarrow x \longrightarrow b \end{array}$$

We first show that a or b again have two different normal forms. With this, we can construct an infinite descending chain of direct successors with different normal forms.

As $\longrightarrow$ is locally confluent, there exists an u with $a \overset{*}{\longrightarrow} u \overset{*}{\longleftarrow} b$.

$$
\begin{tikzcd}
 & & x \arrow[dl] \arrow[dr] & & \\
 & a \arrow[dl, dashed, "*"'] \arrow[dr, dashed, "*"] & \text{LK} & b \arrow[dl, dashed, "*"'] \arrow[dr, dashed, "*"] & \\
\text{NF}_1 & & u & & \text{NF}_2
\end{tikzcd}
$$

Since $\longrightarrow$ terminates by assumption, u has at least one normal form. Case distinction:

$$
\begin{array}{lcl}
u \xrightarrow{*} \text{NF}_1 & : & b \xrightarrow{*} \text{NF}_2 \quad \wedge \quad b \xrightarrow{*} u \xrightarrow{*} \text{NF}_1 \\
 & & \Rightarrow b \text{ has two distinct NFs} \\
u \xrightarrow{*} \text{NF}_2 & : & a \xrightarrow{*} \text{NF}_1 \quad \wedge \quad a \xrightarrow{*} u \xrightarrow{*} \text{NF}_2 \\
 & & \Rightarrow a \text{ has two distinct NFs} \\
\multicolumn{3}{l}{u \xrightarrow{*} \text{NF}_3 \ \wedge \ \text{NF}_1 \neq \text{NF}_3 \neq \text{NF}_2 :} \\
 & & \text{both } a \text{ and } b \text{ have distinct NFs}
\end{array}
$$

□

Homework 42 (Diamand Property)

Let $\longrightarrow$ be a TRS with the following "diamond property":

$$y \longleftarrow x \longrightarrow z \ \wedge \ y \neq z \quad \Longrightarrow \quad \exists u.\, y \longrightarrow u \longleftarrow z$$

Show that, if a has a normal form, all reductions from a to this normal form have the same length.

Homework 43 (Critical Pairs')

Specify all critical pairs of the following term rewriting systems:

a) $f(x, x) \longrightarrow a,\ f(x, g(x)) \longrightarrow b$

b) $f(f(x, y), z) \longrightarrow f(x, f(y, z)),\ f(x, a) \longrightarrow x$

c) $f(f(x, y), z) \longrightarrow f(x, f(y, z)),\ f(a, x) \longrightarrow x$

d) $0 + y \longrightarrow y,\ x + 0 \longrightarrow x,\ s(x) + y \longrightarrow s(x + y),\ x + s(y) \longrightarrow s(x + y)$

Which systems are locally confluent, which are convergent (i.e., terminating and confluent)?

Homework 44 (Completion)

Complete

$$E = \{f(g(f(x))) \approx x\}$$

to a convergent term rewriting system.