

Exercise 39 (Critical Pairs)

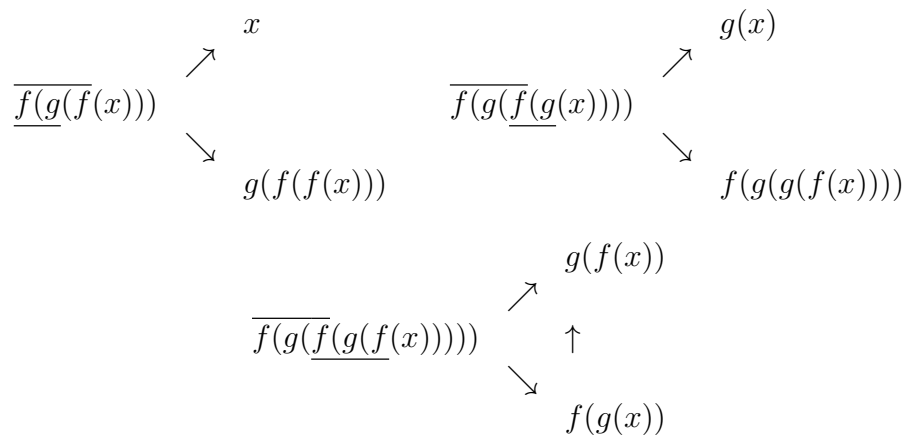
Specify all critical pairs of the following term rewriting systems:

- a) $f(g(f(x))) \rightarrow x, f(g(x)) \rightarrow g(f(x))$
- b) $g(f(x)) \rightarrow f(g^2(h(x))), h(f(x)) \rightarrow f(g(h^2(x))), h(g(x)) \rightarrow g(h(x))$

Which systems are locally confluent, which are convergent (i.e., terminating and confluent)?

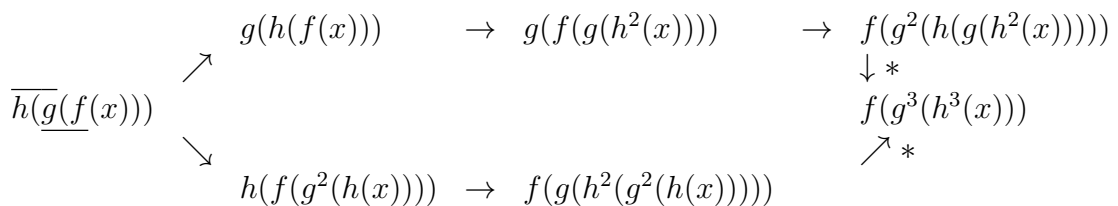
Solution

- a) Three critical pairs:



Not locally confluent, not convergent.

- b) One critical pair:



Locally confluent, terminating, hence convergent.

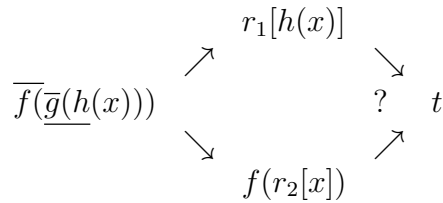
Exercise 40 (Confluence)

Determine terms r_1 and r_2 such that $\{f(g(x)) \rightarrow r_1, g(h(x)) \rightarrow r_2\}$ is confluent.

Solution

- A trivial solution is $r_1 = f(g(x))$ and $r_2 = g(h(x))$. This is obviously confluent, but does not terminate.
- An alternative solution is described in the following:

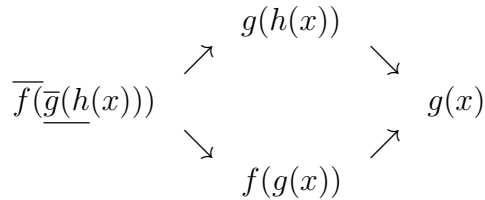
R has a critical pair:



Thus, r_1 and r_2 have to be chosen such that:

$$r_1[h(x)] \xrightarrow{*} t \xleftarrow{*} f(r_2[x])$$

With $r_1 = g(x)$ and $r_2 = g(x)$ we get:



Hence, R is confluent.

Exercise 41 (Newman’s Lemma)

Give an indirect proof of Newman’s Lemma, by showing that \longrightarrow has an infinite reduction sequence, if \longrightarrow is locally confluent but not confluent.

Hint: Show that every element with two distinct normal forms has a direct successor with two distinct normal forms.

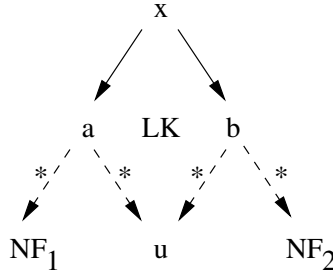
Solution

Proof. Let \longrightarrow be locally confluent but not confluent. We assume that \longrightarrow terminates and derive a contradiction.

$$\begin{aligned}
 \text{not confluent} &\Rightarrow \exists x \text{ with two distinct normal forms } NF_1 \text{ and } NF_2 \\
 &\Rightarrow x \text{ has at least two direct successors: } a \longleftarrow x \longrightarrow b
 \end{aligned}$$

We first show that a or b again have two different normal forms. With this, we can construct an infinite descending chain of direct successors with different normal forms.

As \longrightarrow is locally confluent, there exists an u with $a \xrightarrow{*} u \xleftarrow{*} b$.



Since \longrightarrow terminates by assumption, u has at least one normal form. Case distinction:

$$\begin{aligned}
 u \xrightarrow{*} \text{NF}_1 & : b \xrightarrow{*} \text{NF}_2 \wedge b \xrightarrow{*} u \xrightarrow{*} \text{NF}_1 \\
 & \Rightarrow b \text{ has two distinct NFs} \\
 u \xrightarrow{*} \text{NF}_2 & : a \xrightarrow{*} \text{NF}_1 \wedge a \xrightarrow{*} u \xrightarrow{*} \text{NF}_2 \\
 & \Rightarrow a \text{ has two distinct NFs} \\
 u \xrightarrow{*} \text{NF}_3 \wedge \text{NF}_1 \neq \text{NF}_3 \neq \text{NF}_2 & : \\
 & \text{both } a \text{ and } b \text{ have distinct NFs}
 \end{aligned}$$

□

Homework 42 (Diamond Property)

Let \longrightarrow be a TRS with the following “diamond property”:

$$y \longleftarrow x \longrightarrow z \wedge y \neq z \implies \exists u. y \longrightarrow u \longleftarrow z$$

Show that, if a has a normal form, all reductions from a to this normal form have the same length.

Homework 43 (Critical Pairs')

Specify all critical pairs of the following term rewriting systems:

- $f(x, x) \longrightarrow a, f(x, g(x)) \longrightarrow b$
- $f(f(x, y), z) \longrightarrow f(x, f(y, z)), f(x, a) \longrightarrow x$
- $f(f(x, y), z) \longrightarrow f(x, f(y, z)), f(a, x) \longrightarrow x$
- $0 + y \longrightarrow y, x + 0 \longrightarrow x, s(x) + y \longrightarrow s(x + y), x + s(y) \longrightarrow s(x + y)$

Which systems are locally confluent, which are convergent (i.e., terminating and confluent)?

Homework 44 (Completion)

Complete

$$E = \{f(g(f(x))) \approx x\}$$

to a convergent term rewriting system.