Technische Universität München
Institut für Informatik
Prof. Tobias Nipkow, Ph.D.
Johannes Hölzl and Lars Hupel

Equational Logic
Summer Term 2015
Solutions to Exercise Sheet 9

Exercise 45 (Linear Term Rewriting Systems)

A rewrite rule $l \longrightarrow r$ is called *left-linear* if every variable in l occurs exactly once. Similarly, $l \longrightarrow r$ is called *right-linear* if every variable in r occurs exactly once. A rule is *linear* if it is both right- and left-linear. We say that a term rewriting system is *linear* if it contains only linear rules.

Show:

a) Every linear term rewriting system R that has no critical pairs is confluent. Give a self-contained proof; do not simply apply Corollary 6.3.11 from the book!

   Hint: Show that R is strongly confluent.

b) If R is a linear term rewriting system, and for every critical pair (t_1, t_2) there exists t_0 such that $t_1 \xrightarrow{=} t_0 \xleftarrow{=} t_2$, then R is confluent.

   Hint: Extend the proof of the previous statement.

Solution

a) *Proof.* We show that for any two rules

$$l_1 \to r_1 \tag{1}$$
$$l_2 \to r_2 \tag{2}$$

   from R, we have:

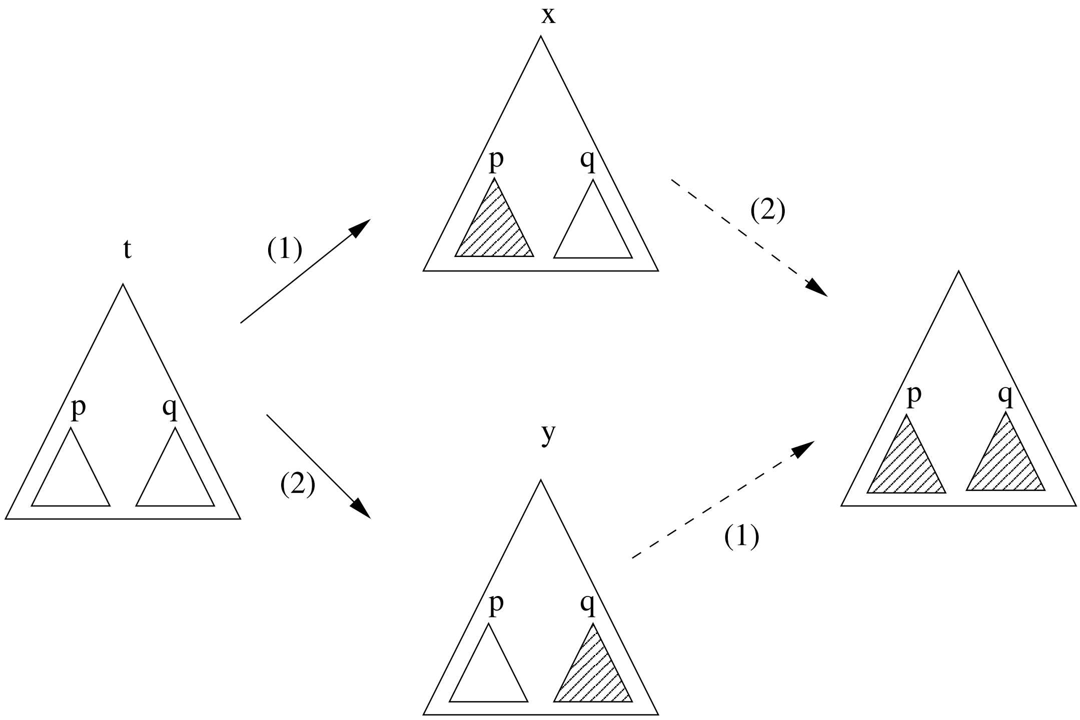
$$x \overset{(1)}{\leftarrow} t \overset{(2)}{\rightarrow} y \;\Rightarrow\; \exists z.\, x \overset{=}{\rightarrow} z \overset{=}{\leftarrow} y$$

   Hence, R is strongly confluent, and thus also confluent.

   Let p be the application position of rule (1) and q the application position of rule (2) in a term t.

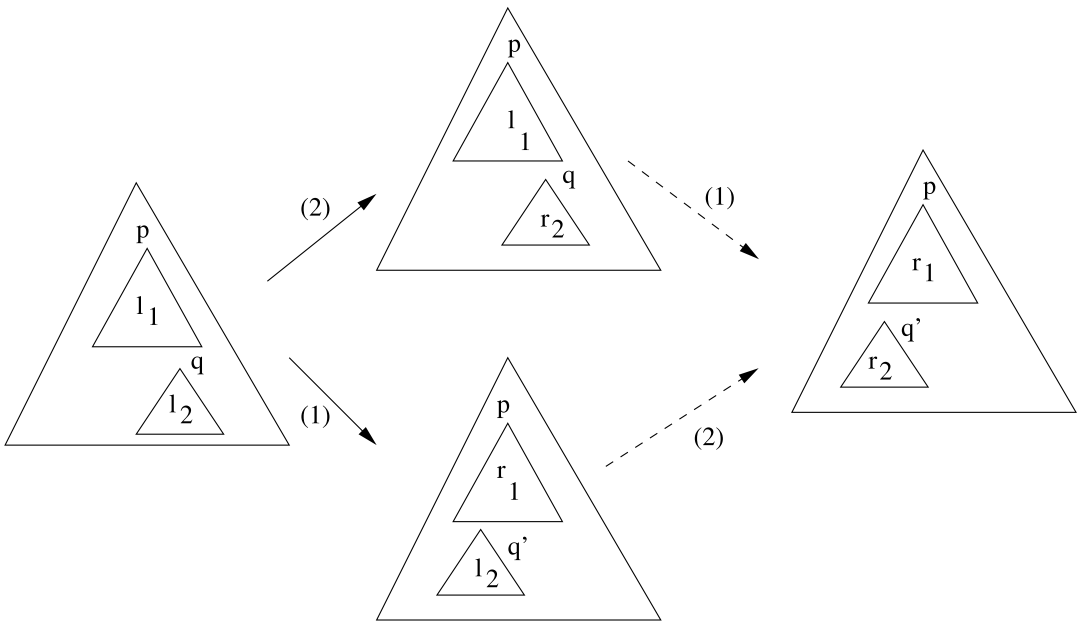
   We have either $p \parallel q$, $p \geq q$ or $q \geq p$. Case distinction:

   - The case $p \parallel q$ is easy.

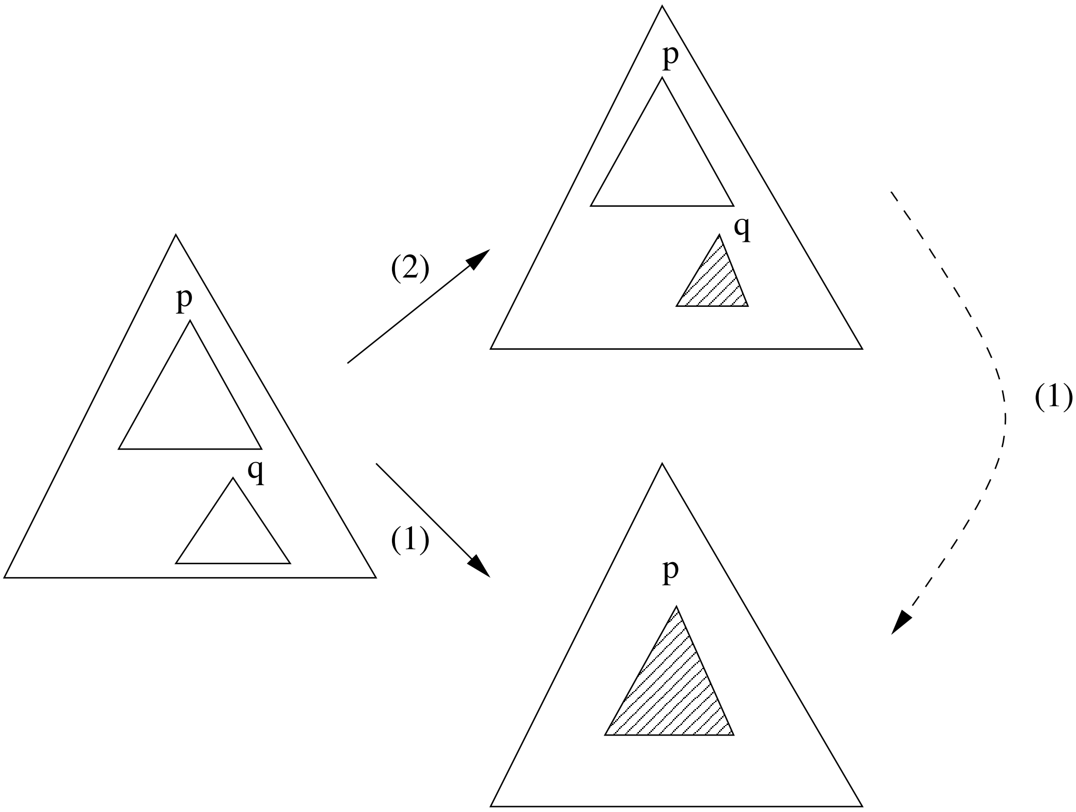


- $q \geq p$. As R has no critical pairs, the position q is below the range that is matched by l_1 in t.

  As R is linear, and in particular, all right and left hand sides contain any variable at most once, the application of a rule $l_1 \rightarrow r_1$ can move the application position of rule $l_2 \rightarrow r_2$ to at most one q'.



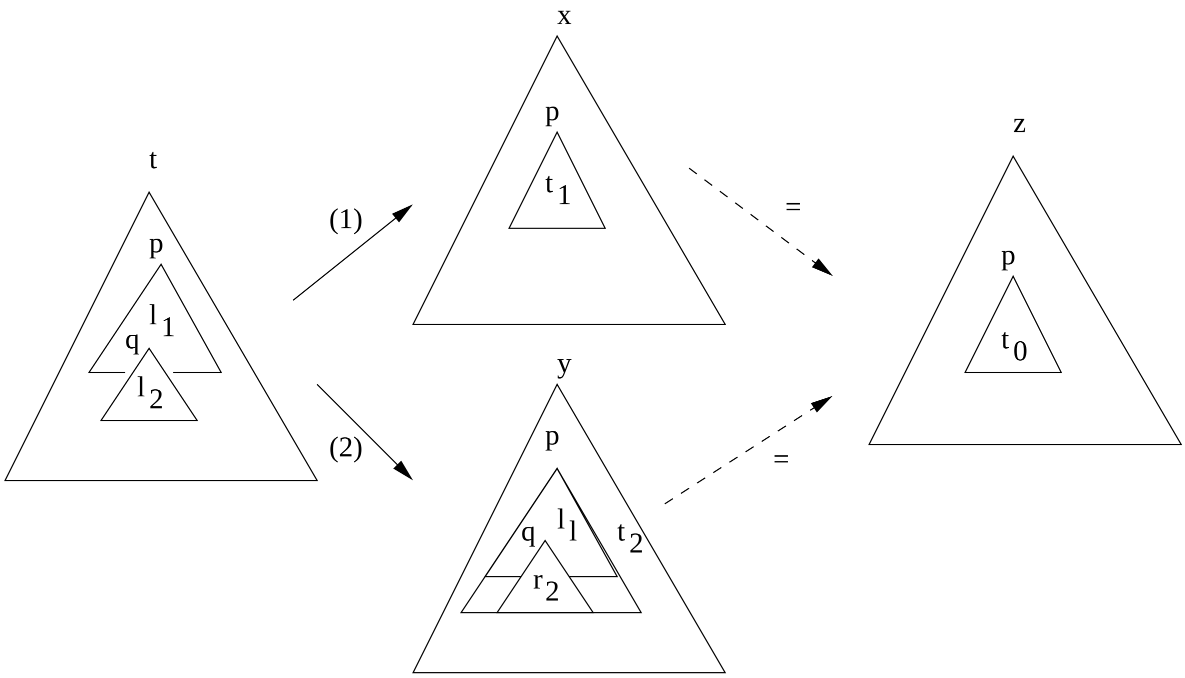
  If r_1 does not contain the respective variable, the rule $l_2 \rightarrow r_2$ cannot be applied. But we then have:



- $p \geq q$. Symmetric.

$\square$

b) Additionally to the cases from the previous part, we have to regard the case $p < q$ for a critical pair. By assumption, we have for each critical pair (t_1, t_2) a t_0 with $t_1 \overset{=}{\to} t_0 \overset{=}{\leftarrow} t_2$:



Exercise 46 (λ-Terms)

Evaluate the following substitutions:

a) $(\lambda y.x(\lambda x.x))\ [(\lambda y.xy)/x]$

b) $(y(\lambda v.xv))\ [(\lambda y.vy)/x]$

Rewrite the following terms such that they are completely parenthesized and conform to the grammar for the λ-calculus given in the lecture (without any shortcut notations).

c) $ux(yz)(\lambda v.vy)$

d) $(\lambda xyz.xz(yz))uvw$

Rewrite the following terms such that there are as few parentheses as possible, and apply all shortcut notation from the lecture:

e) $((u(\lambda x.\ (v(wx))))x)$

f) $(((w(\lambda x.(\lambda y.(\lambda z.((xz)(yz))))))u)v)$

Solution

a) $\lambda y'.(\lambda y.xy)(\lambda x.x)$

b) $y(\lambda z.(\lambda y.vy)z)$

c) $(((ux)(yz))(\lambda v.(vy)))$

d) $((((\lambda x.(\lambda y.(\lambda z.((xz)(yz)))))u)v)w)$

e) $u(\lambda x.\ v(wx))x$

f) $w(\lambda xyz.xz(yz))uv$

Exercise 47 (Formalizations with λ-Terms)

Express the following propositions as λ-terms. Use the constant D as a derivative operator.

a) The derivative of x^2 is $2x$.

b) The derivative of x^2 at 3 is 6.

c) Let f be a function, and let g be defined as $g(x) := f(x^2)$. The derivative of g at x is different from the derivative of f at x^2.

d) Formulate the proposition c) without using the auxiliary function symbol g.

Solution

a) $\mathrm{D}(\lambda x.x^2) = \lambda x.2x$

b) $(\mathrm{D}(\lambda x.x^2))\ 3 = 6$

c) $(\mathrm{D}\,g)x \neq (\mathrm{D}\,f)(x^2)$

d) $(\mathrm{D}(\lambda x.f(x^2)))x \neq (\mathrm{D}\,f)(x^2)$

Homework 48 (Strong Confluence)

Let $\leadsto$ be a relation with $\xrightarrow{=} \subseteq \leadsto \subseteq \xrightarrow{*}$.

Show that $\leadsto$ is strongly confluent iff $\forall t_1 t_2 s.\ t_1 \leftarrow s \leadsto t_2 \implies \exists t.\ t_1 \leadsto t \xleftarrow{*} t_2$.

(Strong confluence of $\rightarrow$ is $\forall t_1 t_2 s.\ t_1 \leftarrow s \rightarrow t_2 \implies \exists t.\ t_1 \xrightarrow{*} t \xleftarrow{=} t_2$)

Homework 49 (Confluence)

Let R be the following term rewriting system:

$$\{f(x,x) \longrightarrow a,\ c \longrightarrow g(c),\ g(x) \longrightarrow f(x, g(x))\}$$

Is R confluent? Justify your answer.

Homework 50 (Substitution Lemma)

Show that, given $x \neq y$ and $x \notin \mathrm{FV}(u)$:

$$s[t/x][u/y] = s[u/y][t[u/y]/x]$$