

Exercise 51 (Confluence of β -Reduction)

In the lecture we have shown the confluence of \longrightarrow_β using the diamond property of parallel β -reduction. In this exercise, we develop an alternative proof.

We define the operation $*$ on λ -terms inductively over the structure of terms:

$$\begin{aligned} x^* &= x \\ (\lambda x. t)^* &= \lambda x. t^* \\ (t_1 t_2)^* &= t_1^* t_2^* \quad \text{if } t_1 t_2 \text{ is not } \beta\text{-reducible.} \\ ((\lambda x. t_1) t_2)^* &= t_1^* [t_2^*/x] \end{aligned}$$

- a) Show that we have for two arbitrary λ -terms s and t : $s > t \implies t > s^*$
 b) Show that \longrightarrow_β is confluent.

Solution

- a) The proof of

$$s > t \implies t > s^*$$

is done by structural induction on s .

1st case: $s = x$.

Let $x > t$. Hence $t = x$. Thus $t = x > x = x^* = s^*$.

2nd case: $s = \lambda x. s_1$.

Let $\lambda x. s_1 > t$. Hence $t = \lambda x. t_1$ for an appropriate t_1 that can be obtained from the inductive definition of $>$, and we have $s_1 > t_1$. As s_1 is a subterm of s , the induction hypothesis implies $t_1 > s_1^*$. Thus we get $\lambda x. t_1 > \lambda x. s_1^* = s^*$.

3rd case: $s = s_1 s_2$, but no β -redex.

Let $s_1 s_2 > t$. Then $t = t_1 t_2$ with $s_1 > t_1$ and $s_2 > t_2$ (again by the inductive definition of $>$) and the induction hypothesis implies $t_1 t_2 > s_1^* s_2^* = s^*$.

4th case: $s = (\lambda x. s_1) s_2$.

Let $(\lambda x. s_1) s_2 > t$. Then there is t_1, t_2 with $s_1 > t_1$ and $s_2 > t_2$ and we have $t = (\lambda x. t_1) t_2$ or $t = t_1 [t_2/x]$, depending on the rule used to derive $s > t$.

Case 4.1 $t = (\lambda x. t_1) t_2$.

By induction hypothesis, we get $t > s_1^* [s_2^*/x] = s^*$.

Case 4.2 $t = t_1[t_2/x]$.

By induction hypothesis, and the substitution property of $>$ (Lemma 1.2.13), we get $t = t_1[t_2/x] > s_1^*[s_2^*/x] = s^*$.

b) Part a) immediately implies the diamond property of $>$: Let $s > t_1$ and $s > t_2$. Then we have both, $t_1 > s^*$ and $t_2 > s^*$.

Exercise 52 (α -Equivalence)

Prove transitivity for α -equivalence, that is, for all terms t_1, t_2, t_3 ,

$$t_1 =_\alpha t_2 \implies t_2 =_\alpha t_3 \implies t_1 =_\alpha t_3$$

Solution

The proof proceeds by induction over $=_\alpha$, using the first premise $t_1 =_\alpha t_2$.

Homework 53 (Parallel β -Reduction)

In the lecture, we defined parallel β -reduction $>$ inductively as follows:

$$\begin{aligned} s > s' &\implies \lambda x.s > \lambda x.s' \\ s > s' \wedge t > t' &\implies (s t) > (s' t') \\ s > s' \wedge t > t' &\implies (\lambda x.s) t > s' [t'/x] \end{aligned}$$

We also showed $> \subseteq \longrightarrow_\beta^*$. In this exercise, you have to show: $\longrightarrow_\beta \subseteq >$

Hint: Use induction over the inductive definition of \longrightarrow_β .

Homework 54 (Expanding lets)

We have a language with **let**-expressions, i.e.:

$$t = v \mid t t \mid \mathbf{let} \ v = t \ \mathbf{in} \ t$$

Write a program which expands all **let**-expressions, the **let**-semantics are:

$$(\mathbf{let} \ v = t_1 \ \mathbf{in} \ t_2) = (\lambda v. t_2) t_1$$