Logics Exercise		
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SS 2016	Exercise Sheet 6	18.05.2016

Submission of Homework: Before tutorial on May 25

Exercise 6.1. [Decidable Theories]
Let S be a set of sentences (= closed formulas) such that S is closed under consequence: if $S \models F$ and F is closed, then $F \in S$. Additionally, assume that S is finitely axiomatizable and complete, i.e. $F \in S$ or $\neg F \in S$ for any sentence F.

1. Give a procedure for deciding wether $S \models F$ for a sentence F.

2. Can you obtain a similar result when the assumption is that the axiom system is only *recursively enumerable*?

Exercise 6.2. [Models of the $\exists^*\forall^*$ Class]
Consider the $\exists^*\forall^*$ class, i.e. formulas of the form

$$\exists x_1 \dots \exists x_n \; \forall y_1 \dots \forall y_m \; F$$

where F is quantifier-free and contains no function symbols. Show that such a formula has a model iff it has a model of size n (assuming $n \geq 1$). What happens if we allow equality in F?

Exercise 6.3. [Ackermann Reduction]
Consider the fragment of (closed) formulas of the form $\forall x_1 \dots \forall x_n \; F$ where F involves no predicates besides equality but arbitrary function symbols. We want to study the *Ackermann reduction*, which yields a decision procedure for this class of formulas. For instance, let

$$F = (x_1 = x_2 \rightarrow f(f(x_1)) = f(g(x_2)))$$

We index the occurrences of each function symbol from the inside out

$$x_1 = x_2 \rightarrow \overbrace{f(\underbrace{f(x_1)}_{f_1})}^{f_2} = \overbrace{f(\underbrace{g(x_2)}_{g_1})}^{f_3}$$

and introduce a fresh variable for each instance. We add constraints which capture the congruence properties for all function symbols involved, and replace terms in the original formula by variables. This yields:

$$\begin{aligned}
&(x_1 = x_{f_1} \rightarrow x_{f_1} = x_{f_2} \wedge \\
&x_{f_1} = x_{g_1} \rightarrow x_{f_2} = x_{f_3} \wedge \\
&x_1 = x_{g_1} \rightarrow x_{f_1} = x_{f_3}) \;\rightarrow \\
&(x_1 = x_2 \rightarrow x_{f_2} = x_{f_3})
\end{aligned}$$

1. Explain how this construction can be used to obtain a procedure for deciding *validity* of formulas from the given fragment.

2. Give a formal description of the reduction.

3. Prove correctness of the Ackermann reduction step in your decision procedure.

Homework 6.1. [**Monadic FOL**] (5 points)
Show that deciding unsatisfiability of monadic FOL formulas can be reduced to deciding unsatisfiability of formulas from the $\exists^*\forall^*$ fragment. Use miniscoping!

Homework 6.2. [$\exists^*\forall^*$ **With Equality**] (5 points)
Show that unsatisfiability of formulas from the $\exists^*\forall^*$ fragment with equality is decidable. Hint: Reduce it to the $\exists^*\forall^*$-fragment without equality.

Homework 6.3. [$\exists^*\forall^2\exists^*$] (5 points)
Show how to reduce deciding unsatisfiability of formulas from the $\exists^*\forall^2\exists^*$-fragment to deciding unsatisfiability of formulas from the $\forall^2\exists^*$-fragment.

Homework 6.4. [**Universal Closure**] (5 points)
Let F be a formula, and $\{x_1, \ldots, x_n\}$ the free variables in F. We define the *universal closure* of F by $\forall F := \forall x_1 \ldots \forall x_n F$.

Let S be a set of closed formulas, and F be a formula. Show that $S \models F$ iff $S \models \forall F$.

Is it also true that $S \models F$ iff $S \models \exists F$, where $\exists F$ is defined analogously to $\forall F$. Proof or counterexample!