Logics Exercise

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Exercise 6.1. **[Decidable Theories]**
Let S be a set of sentences (= closed formulas) such that S is closed under consequence: if $S \models F$ and F is closed, then $F \in S$. Additionally, assume that S is finitely axiomatizable and complete, i.e. $F \in S$ or $\neg F \in S$ for any sentence F.

1. Give a procedure for deciding wether $S \models F$ for a sentence F.
2. Can you obtain a similar result when the assumption is that the axiom system is only *recursively enumerable*?

Exercise 6.2. **[Models of the $\exists^*\forall^*$ Class]**
Consider the $\exists^*\forall^*$ class, i.e. formulas of the form

$$\exists x_1 \dots \exists x_n \, \forall y_1 \dots \forall y_m \; F$$

where F is quantifier-free and contains no function symbols. Show that such a formula has a model iff it has a model of size n (assuming $n \geq 1$). What happens if we allow equality in F?

Exercise 6.3. **[Ackermann Reduction]**
Consider the fragment of (closed) formulas of the form $\forall x_1 \dots \forall x_n \; F$ where F involves no predicates besides equality but arbitrary function symbols. We want to study the *Ackermann reduction*, which yields a decision procedure for this class of formulas. For instance, let

$$F = (x_1 = x_2 \to f(f(x_1)) = f(g(x_2)))$$

We index the occurrences of each function symbol from the inside out

$$x_1 = x_2 \to \overbrace{f(\underbrace{f(x_1)}_{f_1})}^{f_2} = \overbrace{f(\underbrace{g(x_2)}_{g_1})}^{f_3}$$

and introduce a fresh variable for each instance. We add constraints which capture the congruence properties for all function symbols involved, and replace terms in the original formula by variables. This yields:

$$\begin{aligned}
&(x_1 = x_{f_1} \to x_{f_1} = x_{f_2} \land \\
&x_{f_1} = x_{g_1} \to x_{f_2} = x_{f_3} \land \\
&x_1 = x_{g_1} \to x_{f_1} = x_{f_3}) \;\to \\
&(x_1 = x_2 \to x_{f_2} = x_{f_3})
\end{aligned}$$

1. Explain how this construction can be used to obtain a procedure for deciding *validity* of formulas from the given fragment.
2. Give a formal description of the reduction.
3. Prove correctness of the Ackermann reduction step in your decision procedure.

Homework 6.1. [**Monadic FOL**] (5 points)
Show that deciding unsatisfiability of monadic FOL formulas can be reduced to deciding unsatisfiability of formulas from the $\exists^*\forall^*$ fragment. Use miniscoping!

Solution: Due to slide decision-problem/6, it's enough to show that after miniscoping no nested quantifiers remain.

We prove, by induction on the structure of the formula, that after miniscoping, for each sub-formula of the form $\forall x.F$ resp. $\exists x.F$, F is a disjunction resp. conjunction of literals, each literal containing x free.

The only interesting cases are the quantifier cases. Assume we have a formula of the form $\exists x.F$, such that no miniscoping rules are applicable, and by induction hypothesis, below quantifiers in F there are only disjunctions/conjunctions of literals containing the bound variable.

As no miniscoping rules are applicable, F must be a conjunction of literals and quantified formulas, such that each conjunct contains x free. So assume F contains a quantified formula, i.e., $F = \ldots \wedge Qy.F' \wedge \ldots$. By induction hypothesis, F' is a disjunction/conjunction of literals, each literal containing y free. However, as we are in the monadic fragment, a literal can contain at most one free variable. Thus, F' cannot contain x free, which is a contradiction to F containing quantifiers. Thus, F only contains literals, and thus has the desired shape.

The case for $\forall x.F$ is analogously. qed.

Homework 6.2. [$\exists^*\forall^*$ **With Equality**] (5 points)
Show that unsatisfiability of formulas from the $\exists^*\forall^*$ fragment with equality is decidable. Hint: Reduce it to the $\exists^*\forall^*$-fragment without equality.

Solution: Applying the reduction of equality to non-equality from the lecture only inserts some (isolated) $\forall$-quantifiers, thus preserving the $\exists^*\forall^*$-fragment.

Homework 6.3. [$\exists^*\forall^2\exists^*$] (5 points)
Show how to reduce deciding unsatisfiability of formulas from the $\exists^*\forall^2\exists^*$-fragment to deciding unsatisfiability of formulas from the $\forall^2\exists^*$-fragment.

Solution: Using skolemization for the outer existential quantifiers preserves satisfiability, and replaces variables by skolem constants, i.e., introduces no function symbols of arity > 0. The resulting formula is obviously in the $\forall^2\exists^*$-fragment.

Homework 6.4. **[Universal Closure]** (5 points)
Let F be a formula, and $\{x_1, \ldots, x_n\}$ the free variables in F. We define the *universal closure* of F by $\forall F := \forall x_1 \ldots \forall x_n F$.

Let S be a set of closed formulas, and F be a formula. Show that $S \models F$ iff $S \models \forall F$.

Is it also true that $S \models F$ iff $S \models \exists F$, where $\exists F$ is defined analogously to $\forall F$. Proof or counterexample!

Solution: As S is closed, in any model of S, the bindings for the free variables of F can be changed arbitrarily without changing the model property. This implies the $\implies$ direction of the first part. The reverse direction is trivial.

The second part does not hold, consider, for example $P(c) \models \exists x(P(x) \land P(c))$, which does hold, as x can always be chosen to have the same value as c. However, $P(c) \models P(x) \land P(c)$ does not hold, as x may be bound to some value for which P does not hold.