

LOGICS EXERCISE

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EXERCISE SHEET 12

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Exercise 12.1. [Resolution and Left-Sided Sequent Calculus]

Consider the left-sided sequent calculus from homework 11.4. Show that every proof tree in the left-sided sequent calculus can be transformed into a resolution proof and vice versa (for propositional logic).

Hint: For the direction from left-sided sequent calculus to resolution you can simplify work by assuming that the sequents only consist of sets of clauses. How can you reduce the number of sequent rules you have to consider in this case?

Exercise 12.2. [From Natural Deduction to Hilbert Calculus]

First prove the following formula with natural deduction:

$$(F \wedge G) \rightarrow (G \wedge F)$$

Now transform the resulting proof tree to a linear proof or proof tree in Hilbert calculus.

Exercise 12.3. [A Smaller Set of Axioms for Hilbert Calculus]

In the lecture, Hilbert calculus for propositional logic was introduced by means of nine axioms. However, the following three axioms are already sufficient (assuming that \wedge and \vee are derived connectives):

- A1: $F \rightarrow (G \rightarrow F)$
- A2: $(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$
- A10: $(\neg F \rightarrow \neg G) \rightarrow (G \rightarrow F)$

(Axioms A1 and A2 already appeared in the lecture). Derive the following axioms from A1, A2 and A10 with help of \rightarrow_E :

- $\neg(F \rightarrow F) \rightarrow G$
- $\neg\neg F \rightarrow F$

where

$$\begin{aligned}\Gamma &= \forall x \exists y \forall z (\neg P(x, z) \wedge P(z, y)) \\ \Delta &= \forall z (\neg P(c, z) \wedge P(z, y))\end{aligned}$$

(*): This step works on the quantifier hidden in Δ .

Homework 12.2. [Sequent Calculus (II)] (5 points)

Prove that $\vdash_G \Gamma \Rightarrow \Delta$ implies $\vdash_G \Gamma[t/x] \Rightarrow \Delta[t/x]$, where, for a set of formulas Γ , we define $\Gamma[t/x]$ to be $\{F[t/x] \mid F \in \Gamma\}$, i.e. free occurrences of x are replaced by t . Give two different proofs:

1. A syntactic proof, transforming the proof tree of $\vdash_G \Gamma \Rightarrow \Delta$.
2. A semantic proof, using correctness and completeness of \vdash_G .

Solution:

1. Given a proof of $\Gamma \Rightarrow \Delta$, we can apply the substitution $x \mapsto p$ throughout the proof tree and obtain a valid proof of $\Gamma[x \mapsto p] \Rightarrow \Delta[x \mapsto p]$. Formally, this is an induction on the structure of the proof. Each case in the induction considers one proof rule or axiom, and we have to show that the new proof is still an instance of the axiom. This is trivial in all cases, since the rules are given as schemas that do only concern the structure of the formulas, which is not changed by the substitution.
2. We can also prove the claim by using soundness and completeness. Assume that $\Gamma = \{p_1, \dots, p_n\}$, $\Delta = \{q_1, \dots, q_m\}$ and $\Gamma \Rightarrow \Delta$ is derivable. By soundness we know that $p_1 \wedge \dots \wedge p_n \rightarrow q_1 \vee \dots \vee q_m$ is a tautology. By the substitution lemma (Corollary 2.4) we know that substituting p for x yields a tautology again, and the completeness theorem asserts that the resulting sequent is provable.

Homework 12.3. [Marriage Problem] (5 points)

Use the compactness theorem to prove the following: Let B be an infinite set of boys, such that each boy has a finite number of girlfriends. Moreover, any set of k boys has at least k different girlfriends. Show that each boy can marry one of his girlfriends, where polygamy is forbidden.

Hint: Find an (infinite) set of formulas that is satisfiable if and only if a marriage matching exists. Be careful not to form conjunctions or disjunctions over infinite sets!

Solution: For each $b \in B$, we write G_b for the finite set of girlfriends of b . We use atom M_{bg} to denote that b marries g . We say that a set of boys is *consistent*, if every boy can marry one of his girlfriends without any of them committing bigamy.

We have to prove that B is consistent if all finite subsets of B are. To this aim, we define the set of formulas $\text{Con}(A)$ of a set $A \subseteq B$ that expresses consistency:

$$\begin{aligned} \text{Con}(A) = & \left\{ \bigvee_{g \in G_b} M_{bg} \mid b \in A \right\} \\ & \cup \left\{ \neg(M_{b_1g} \wedge M_{b_2g}) \mid b_1, b_2 \in A, g \in G_{b_1} \cap G_{b_2} \right\} \end{aligned}$$

Due to compactness, $\text{Con}(B)$ is satisfiable when any finite subset of it is satisfiable. So take a finite set $F \subseteq \text{Con}(B)$, then we can take the set A of boys that is mentioned in some formula in F , and $F \subseteq \text{Con}(A)$. But $\text{Con}(A)$ is satisfiable by assumption, and hence F is satisfiable, too.

Note that when encoding some constraints into a propositional formula using the big \bigwedge and \bigvee notation, we must make sure that the sets involved are finite. For example,

$$\bigwedge_{\substack{b_1, b_2 \in B \\ g \in G_{b_1}}} \neg(M_{b_1g} \wedge M_{b_2g})$$

is not a propositional formula, since formulas must be finite.

Homework 12.4. [Size of Resolution Proofs] (5 points)

Find a valid propositional logic formula, such that each sequent calculus proof tree has at least quadratic size in the size of the formula, and the formula has an at most linear size resolution proof.

Solution: Only the rules $\vee L$, $\rightarrow R$, and $\wedge R$ lead to a proof explosion in the sequent calculus. Thus, for example, the tautology $p = (A \vee A \vee \dots \vee A \rightarrow A \wedge A \wedge \dots \wedge A)$ causes a proof quadratic in the size of p . Intuitively, every A on the right hand side is deduced from every A on the left-hand side.

Resolution performs much better on p . After conversion to CNF, we obtain several clauses of the form $[\neg A]$ and $[A]$, which can be resolved to the empty clause by just one step.