

LOGICS EXERCISE

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EXERCISE SHEET 13

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Exercise 13.1. [Relations]

Use natural deduction to show: Let T be a total relation, and A be an antisymmetric relation, and $T \subseteq A$, then $A \subseteq T$.

1. Express the above statement in first-order logic
2. Prove it by natural deduction

Exercise 13.2. [Proof!?!]

Consider the following “proof” of an invalid sequent:

$$\frac{\overline{\exists yP(y, y), \forall x\exists yP(x, y) \Rightarrow \exists yP(y, y)} \text{ axiom}}{\forall x\exists yP(x, y) \Rightarrow \exists yP(y, y)} \forall L \text{ with } [y/x]$$

What went wrong?

Exercise 13.3. [Translation from Sequent Calculus to Natural Deduction]

Show that $\vdash_G \Gamma \Rightarrow \Delta$ implies $\Gamma \vdash_N \bigvee \Delta$. Only consider the cases for axioms and \wedge .

Exercise 13.4. [Natural Deduction with Equality]

Prove by natural deduction:

1. $f^3(a) = a \wedge f^5(a) = a \rightarrow f(a) = a$
2. $A[t/x] \rightarrow \forall x(x = t \rightarrow A)$ if x does not occur free in t

Homework 13.1. [Hintikka Sets in Predicate Logic]

(6 points)

A Hintikka Set (of predicate logic) for a signature Σ is a set H of formulae, such that:

1. For any atom A , not both $A \in H$ and $\neg A \in H$
2. If $\neg\neg Z \in H$ then also $Z \in H$
3. If $F_1 \wedge F_2 \in H$ then also $F_1 \in H$ and $F_2 \in H$
4. If $\neg(F_1 \vee F_2) \in H$ then also $\neg F_1 \in H$ and $\neg F_2 \in H$
5. If $F_1 \vee F_2 \in H$ then also $F_1 \in H$ or $F_2 \in H$
6. If $\neg(F_1 \wedge F_2) \in H$ then also $\neg F_1 \in H$ or $\neg F_2 \in H$
7. If $(\forall x F) \in S$, then $F[t/x] \in S$ for all Σ -terms t
8. If $(\exists x F) \in S$, then $F[t/x] \in S$ for some Σ -term t
9. If $\neg(\forall x F) \in S$, then $(\exists x \neg F) \in S$
10. If $\neg(\exists x F) \in S$, then $(\forall x \neg F) \in S$

Show: Every Hintikka-set of predicate logic is satisfiable.

Solution: We choose $U_{\mathcal{A}}$ as the set of all Σ -terms. For any function symbol f and Σ -terms t_1, \dots, t_n , let $I_{\mathcal{A}}(f(t_1, \dots, t_n)) = f(t_1, \dots, t_n)$. For any predicate symbol P and Σ -terms t_1, \dots, t_n , let $I_{\mathcal{A}}(P(t_1, \dots, t_n)) = 1$ iff $P(t_1, \dots, t_n) \in H$. Finally, for any free variable x , we set $I_{\mathcal{A}}(x) = x$ (assuming that the symbols for free variables are distinct from Σ). Then we show by induction on the size of the formula, that for all $F \in H$, we have $\mathcal{A}(F) = 1$.

Homework 13.2. [Proofs with Sequent Calculus for Predicate Logic] (5 points)
 Prove the following formulas in the sequent calculus or construct a counterexample from a stuck proof attempt in the case a formula does not hold:

- $\exists x(P \rightarrow Q(x)) \rightarrow (P \rightarrow \exists z(Q(z)))$
- $\exists x(P \rightarrow Q(x)) \rightarrow (P \rightarrow \forall z(Q(z)))$

Solution:

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$$\frac{\frac{\frac{Q(y_1), P \Rightarrow Q(y_1), \exists z(Q(z))}{P \Rightarrow P, \exists z(Q(z))} \exists R}{Q(y_1), P \Rightarrow \exists z(Q(z))} \rightarrow L}{\frac{P, P \rightarrow Q(y_1) \Rightarrow \exists z(Q(z))}{P \rightarrow Q(y_1) \Rightarrow P \rightarrow \exists z(Q(z))} \rightarrow R}{\frac{\exists x(P \rightarrow Q(x)) \Rightarrow P \rightarrow \exists z(Q(z))}{\Rightarrow \exists x(P \rightarrow Q(x)) \rightarrow (P \rightarrow \exists z(Q(z)))} \exists L} \rightarrow R$$

•

$$\frac{\frac{\frac{Q(y_1), P \Rightarrow Q(y_2)}{P \Rightarrow P, \forall z(Q(z))} \forall R}{Q(y_1), P \Rightarrow \forall z(Q(z))} \rightarrow L}{\frac{P, P \rightarrow Q(y_1) \Rightarrow \forall z(Q(z))}{P \rightarrow Q(y_1) \Rightarrow P \rightarrow \forall z(Q(z))} \rightarrow R}{\frac{\exists x(P \rightarrow Q(x)) \Rightarrow P \rightarrow \forall z(Q(z))}{\Rightarrow \exists x(P \rightarrow Q(x)) \rightarrow (P \rightarrow \forall z(Q(z)))} \exists L} \rightarrow R$$

A counterexample can be constructed from where we got stuck by setting $U_{\mathcal{A}} = \{y_1, y_2\}$, $I_{\mathcal{A}}(Q(y_1)) = 1$, $I_{\mathcal{A}}(P) = 1$, and $I_{\mathcal{A}}(Q(y_2)) = 0$.

Homework 13.3. [Natural Deduction and Resolution in Predicate Logic] (5 points)
 Show that the following formula is valid – once with natural deduction and once with resolution:

$$\neg(\forall x(\exists y(\neg P(x) \wedge P(y))))$$

Solution:

$$\frac{\frac{\frac{[\forall x(\exists y(\neg P(x) \wedge P(y)))]^1}{\exists y(\neg P(x_1) \wedge P(y))} \forall E}{\neg P(x_1) \wedge P(y_1)} \exists E}{P(y_1)} \wedge E_2 \quad \frac{\frac{[\forall x(\exists y(\neg P(x) \wedge P(y)))]^1}{\exists y(\neg P(y_1) \wedge P(y))} \forall E}{\neg P(y_1) \wedge P(y_2)} \exists E}{\neg P(y_1)} \wedge E_1}{\perp} \neg E \quad \frac{}{\neg(\forall x(\exists y(\neg P(x) \wedge P(y))))} \neg I (1)$$

Note that a proof step of the following form would **not** be allowed:

$$\frac{\forall x(\exists y(\neg P(x) \wedge P(y)))}{\exists y(\neg P(y) \wedge P(y))} \forall E$$

For the resolution proof, we first need to transform the formula's negation to Skolem form:

$$\forall x(\neg P(x) \wedge P(f(x)))$$

Then it is easy to find a refutation:

1. $\neg P(y)$
2. $P(f(x))$
3. \square from 1, 2 with $[f(x)/y]$

Homework 13.4. [Left-Sided Sequent Calculus for Predicate Logic] (4 points)
 Extend the left-sided sequent calculus for propositional logic from exercise 11.4 with the necessary rules for predicate logic and repeat the correctness proof for the new rules.

Solution: The new left rules:

$$\frac{F[t/x], \forall x F, \Gamma \Rightarrow}{\forall x F, \Gamma \Rightarrow} \forall L' \qquad \frac{F[y/x], \Gamma \Rightarrow}{\exists x F, \Gamma \Rightarrow} \exists L'(*)$$

(*) y not free in F, Γ .

The new right rules:

$$\frac{\neg F[y/x], \Gamma \Rightarrow}{\neg(\forall x F), \Gamma \Rightarrow} \forall R'(*) \qquad \frac{\neg F[t/x], \neg(\exists x F), \Gamma \Rightarrow}{\neg(\exists x F), \Gamma \Rightarrow} \exists R'$$

(*) y not free in F, Γ .

As for the propositional case, the subtle cases arise for the right rules when proving the direction $\Gamma, \neg\Delta \Rightarrow \implies \Gamma \Rightarrow \Delta$. This exemplified for $\forall R'$ below.

Induction step: The induction hypothesis is that for all Γ and Δ if we have a proof of length n for $\Gamma, \neg\Delta \Rightarrow$, then we have a proof for $\Gamma \Rightarrow \Delta$. We are further given a proof for $\Gamma, \neg\Delta \Rightarrow$ of length $n+1$. We consider the last rule used and only look at the two rules for implication:

(Case $\forall R'$) We now have to consider two cases because $\neg(\forall x F)$ might arise from Γ or Δ . The proof can look like:

$$\frac{\begin{array}{c} \vdots \text{ length } n \\ \Gamma, \neg F[y/x], \neg\Delta' \Rightarrow \end{array}}{\Gamma, \neg(\forall x F), \neg\Delta' \Rightarrow} \forall R'$$

Using the induction hypothesis once, we get:

$$\frac{\begin{array}{c} \vdots \\ \Gamma, F[y/x], \Delta' \end{array}}{\Gamma \Rightarrow \forall x F, \Delta'} \forall R$$

And the proof can look like:

$$\frac{\begin{array}{c} \vdots \text{ length } n \\ \Gamma', \neg F[y/x], \neg\Delta \Rightarrow \end{array}}{\Gamma', \neg(\forall x F), \neg\Delta \Rightarrow} \forall R'$$

Using the induction hypothesis and $\neg L$ once, we get:

$$\frac{\begin{array}{c} \vdots \\ \neg F[y/x], \Gamma' \Rightarrow \Delta \end{array}}{\Gamma' \Rightarrow F[y/x], \Delta} \neg R$$

$$\frac{\Gamma' \Rightarrow F[y/x], \Delta}{\Gamma' \Rightarrow \forall x F, \Delta} \forall R$$

$$\frac{\Gamma' \Rightarrow \forall x F, \Delta}{\Gamma', \neg(\forall x F) \Rightarrow \Delta} \neg L$$