

# Propositional Logic

## Basics

# Syntax of propositional logic

## Definition

An **atomic formula** (or **atom**) has the form  $A_i$  where  $i = 1, 2, 3, \dots$

**Formulas** are defined inductively:

- ▶ All atomic formulas are formulas
- ▶ For every formula  $F$ ,  $\neg F$  is a formula.
- ▶ For all formulas  $F$  und  $G$ ,  $(F \wedge G)$  and  $(F \vee G)$  are formulas.

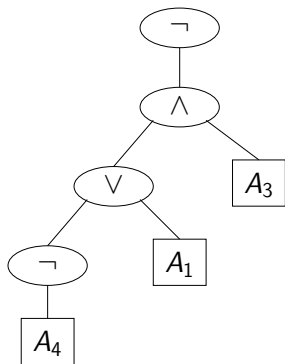
$\neg F$	is called	the <b>negation</b> of $F$
$(F \wedge G)$	is called	the <b>conjunction</b> of $F$ and $G$
$(F \vee G)$	is called	the <b>disjunction</b> of $F$ and $G$

## Syntax tree of a formula

Every formula can be represented by a syntax tree.

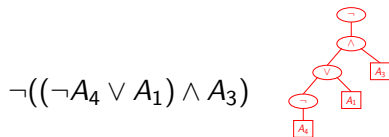
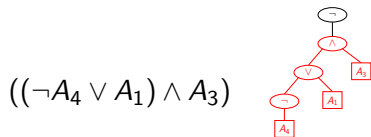
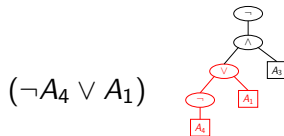
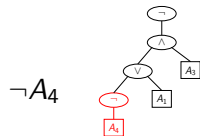
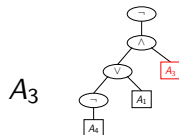
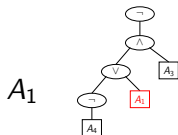
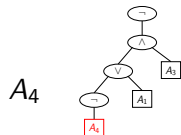
### Example

$$F = \neg((\neg A_4 \vee A_1) \wedge A_3)$$



# Subformulas

The **subformulas** of a formula are the formulas corresponding to the subtrees of its syntax tree.



# Induction on formulas

Proof by induction on the structure of a formula:

In order to prove some property  $\mathcal{P}(F)$  for all formulas  $F$  it suffices to prove the following:

- ▶ Base case:  
prove  $\mathcal{P}(A_i)$  for all atoms  $A_i$
- ▶ Induction step for  $\neg$ :  
prove  $\mathcal{P}(\neg F)$  under the induction hypothesis  $\mathcal{P}(F)$
- ▶ Induction step for  $\wedge$ :  
prove  $\mathcal{P}(F \wedge G)$  under the induction hypotheses  $\mathcal{P}(F)$  and  $\mathcal{P}(G)$
- ▶ Induction step for  $\vee$ :  
prove  $\mathcal{P}(F \vee G)$  under the induction hypotheses  $\mathcal{P}(F)$  and  $\mathcal{P}(G)$

# Semantics of propositional logic (I)

The elements of the set  $\{0, 1\}$  are called **truth values**.  
(You may call 0 “false” and 1 “true”)

An **assignment** is a function  $\mathcal{A} : D \rightarrow \{0, 1\}$ , where  $D$  is any subset of the atoms.

We extend  $\mathcal{A}$  to a function  $\hat{\mathcal{A}} : E \rightarrow \{0, 1\}$ , where  $E \supseteq D$  is the set of formulas that can be built up using only the atoms from  $D$ .

## Semantics of propositional logic (II)

$$\begin{aligned}\hat{\mathcal{A}}(A) &= \mathcal{A}(A) \quad \text{if } A \in D \text{ is an atom} \\ \hat{\mathcal{A}}((F \wedge G)) &= \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 1 \text{ and } \hat{\mathcal{A}}(G) = 1 \\ 0 & \text{otherwise} \end{cases} \\ \hat{\mathcal{A}}((F \vee G)) &= \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 1 \text{ or } \hat{\mathcal{A}}(G) = 1 \\ 0 & \text{otherwise} \end{cases} \\ \hat{\mathcal{A}}(\neg F) &= \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 0 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

We write  $\mathcal{A}$  instead of  $\hat{\mathcal{A}}$ .

# Truth tables (I)

We can compute  $\hat{\mathcal{A}}$  with the help of **truth tables**.

Tables for the operators  $\vee$ ,  $\wedge$ ,  $\neg$ :

$A$	$\vee$	$B$
0	0	0
0	1	1
1	1	0
1	1	1

$A$	$\wedge$	$B$
0	0	0
0	0	1
1	0	0
1	1	1

$\neg$	$A$
1	0
0	1



# Abbreviations

$A, B, C,$   
 $P, Q, R,$  or ...    instead of     $A_1, A_2, A_3 \dots$

$(F_1 \rightarrow F_2)$     abbreviates     $(\neg F_1 \vee F_2)$

$(F_1 \leftrightarrow F_2)$     abbreviates     $((F_1 \wedge F_2) \vee (\neg F_1 \wedge \neg F_2))$

$(\bigvee_{i=1}^n F_i)$     abbreviates     $(\dots((F_1 \vee F_2) \vee F_3) \vee \dots \vee F_n)$

$(\bigwedge_{i=1}^n F_i)$     abbreviates     $(\dots((F_1 \wedge F_2) \wedge F_3) \wedge \dots \wedge F_n)$

$\top$     abbreviates     $(A_1 \vee \neg A_1)$

$\perp$     abbreviates     $(A_1 \wedge \neg A_1)$

## Truth tables (II)

Tables for the operators  $\rightarrow$ ,  $\leftrightarrow$ :

$A$	$\rightarrow$	$B$
0	1	0
0	1	1
1	0	0
1	1	1

Name: **implication**

Truth table:

if  $A$  is 1 then  $B$  is 1

$A$	$\leftrightarrow$	$B$
0	1	0
0	0	1
1	0	0
1	1	1

Name: **bi-implication**

Truth table:

$A$  is 1 if and only if  $B$  is 1

# Models

## Definition

Let  $F$  be a formula and let  $\mathcal{A}$  be an assignment.

$\mathcal{A}$  is **suitable** for  $F$  if it is defined for every atom  $A_i$  occurring in  $F$ .

Let  $\mathcal{A}$  be suitable for  $F$ :

If  $\mathcal{A}(F) = 1$  then we write  $\mathcal{A} \models F$   
and say  $F$  is true under  $\mathcal{A}$   
or  $\mathcal{A}$  is a model of  $F$

If  $\mathcal{A}(F) = 0$  then we write  $\mathcal{A} \not\models F$   
and say  $F$  is false under  $\mathcal{A}$   
or  $\mathcal{A}$  is not a model of  $F$

# Validity and satisfiability

## Definition (Validity)

A formula  $F$  is **valid** (or a **tautology**) if every suitable assignment for  $F$  is a model of  $F$ .

We write  $\models F$  if  $F$  is valid, and  $\not\models F$  otherwise.

## Definition (Satisfiability)

A formula  $F$  is **satisfiable** if it has at least one model, otherwise  $F$  is **unsatisfiable**.

A (finite or infinite!) set of formulas  $S$  is **satisfiable** if there is an assignment that is a model of every formula in  $S$ .

## Exercise

	Valid	Satisfiable	Unsatisfiable
$A$			
$A \vee B$			
$A \vee \neg A$			
$A \wedge \neg A$			
$A \rightarrow \neg A$			
$A \rightarrow (B \rightarrow A)$			
$A \rightarrow (A \rightarrow B)$			
$A \leftrightarrow \neg A$			

## Exercise

Which of the following statements are true?

	Y	C.ex.
If $F$ is valid, then $F$ is satisfiable		
If $F$ is satisfiable, then $\neg F$ is satisfiable		
If $F$ is valid, then $\neg F$ is unsatisfiable		
If $F$ is unsatisfiable, then $\neg F$ is unsatisfiable		

# Mirroring principle

all propositional formulas

valid formulas  $G$	satisfiable but not valid formulas  $F$   $\neg F$	unsatisfiable formulas  $\neg G$
------------------------------	--	---

# Consequence

## Definition

A formula  $G$  is a (semantic) consequence of a set of formulas  $M$  if every model  $\mathcal{A}$  of all  $F \in M$  that is suitable for  $G$  is also a model of  $G$ . Then we write  $M \models G$ .

In a nutshell:

“Every model of  $M$  is a model of  $G$ .”

## Example

$(A \vee B), (A \rightarrow B), ((B \wedge R) \rightarrow \neg A), R \models ((R \wedge \neg A) \wedge B)$



## Exercise

$M$	$F$	$M \models F?$
$A$	$A \vee B$	
$A$	$A \wedge B$	
$A, B$	$A \vee B$	
$A, B$	$A \wedge B$	
$A \wedge B$	$A$	
$A \vee B$	$A$	
$A, A \rightarrow B$	$B$	

# Consequence, validity, satisfiability

## Exercise

*The following statements are equivalent:*

1.  $F_1, \dots, F_k \models G$
2.  $\models ((\bigwedge_{i=1}^k F_i) \rightarrow G)$

## Exercise

*The following statements are equivalent:*

1.  $(F \rightarrow G)$  is valid.
2.  $(F \wedge \neg G)$  is unsatisfiable.

## Exercise

Let  $M$  be a set of formulas, and let  $F$  and  $G$  be formulas.  
Which of the following statements hold?

	Y/N	C.ex.
If $F$ satisfiable then $M \models F$ .		
If $F$ valid then $M \models F$ .		
If $F \in M$ then $M \models F$ .		
If $F \models G$ then $\neg F \models \neg G$ .		

# Warning

The symbol  $\models$  is overloaded:

$$\mathcal{A} \models F$$

$$\models F$$

$$M \models F$$