

Basic Proof Theory

Propositional Logic

Sequent Calculus rules

$$\frac{}{\perp, \Gamma \Rightarrow \Delta} \perp L$$

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta} \neg L$$

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta} \wedge L$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \vee L$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad G, \Gamma \Rightarrow \Delta}{F \rightarrow G, \Gamma \Rightarrow \Delta} \rightarrow L$$

$$\frac{}{F, \Gamma \Rightarrow F, \Delta} Ax$$

$$\frac{F, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg F, \Delta} \neg R$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \wedge G, \Delta} \wedge R$$

$$\frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta} \vee R$$

$$\frac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \rightarrow G, \Delta} \rightarrow R$$

Tableaux Calculus rules

$$\frac{- \neg F}{+ F}$$

$$\frac{+ \neg F}{- F}$$

$$\frac{- F \wedge G}{- F \quad - G}$$

$$\frac{+ F \wedge G}{+ F \mid + G}$$

$$\frac{- F \vee G}{- F \mid - G}$$

$$\frac{+ F \vee G}{+ F \quad + G}$$

$$\frac{- F \rightarrow G}{+ F \mid - G}$$

$$\frac{+ F \rightarrow G}{- F \quad + G}$$

Natural Deduction rules

$$\frac{F \quad G}{F \wedge G} \wedge I$$

$$\frac{F \wedge G}{F} \wedge E_1 \quad \frac{F \wedge G}{G} \wedge E_2$$

$$\frac{\begin{array}{c} [F] \\ \vdots \\ G \end{array}}{F \rightarrow G} \rightarrow I$$

$$\frac{F \rightarrow G \quad F}{G} \rightarrow E$$

$$\frac{F}{F \vee G} \vee I_1 \quad \frac{G}{F \vee G} \vee I_2$$

$$\frac{\begin{array}{c} [F] \quad [G] \\ \vdots \quad \vdots \\ F \vee G \quad H \quad H \end{array}}{H} \vee E$$

$$\frac{\begin{array}{c} [\neg F] \\ \vdots \\ \perp \end{array}}{F} \perp$$

Natural Deduction rules

Rules for \neg are special cases of rules for \rightarrow :

$$\frac{\begin{array}{c} [F] \\ \vdots \\ \perp \end{array}}{\neg F} \neg I \qquad \frac{\neg F \quad F}{\perp} \neg E$$

Hilbert System

The rule $\rightarrow E$ together with the following axioms:

$$F \rightarrow G \rightarrow F \quad (A_1)$$

$$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H \quad (A_2)$$

$$F \rightarrow G \rightarrow F \wedge G \quad (A_3)$$

$$F \wedge G \rightarrow F \quad (A_4)$$

$$F \wedge G \rightarrow G \quad (A_5)$$

$$F \rightarrow F \vee G \quad (A_6)$$

$$G \rightarrow F \vee G \quad (A_7)$$

$$F \vee G \rightarrow (F \rightarrow H) \rightarrow (G \rightarrow H) \rightarrow H \quad (A_8)$$

$$(\neg F \rightarrow \perp) \rightarrow F \quad (A_9)$$

Convention: \rightarrow associates to the right:

$F \rightarrow G \rightarrow H$ means $F \rightarrow (G \rightarrow H)$

Basic Proof Theory

Predicate Logic

Substitution

Substitution $F[t/x]$ assumes that bound variables in F are automatically renamed to avoid capturing free variables in t .

Example

$$(\forall x P(x, y))[f(x)/y] = \forall x' P(x', f(x))$$

Sequent Calculus rules

$$\frac{F[t/x], \forall x F, \Gamma \Rightarrow \Delta}{\forall x F, \Gamma \Rightarrow \Delta} \forall L$$

$$\frac{\Gamma \Rightarrow F[y/x], \Delta}{\Gamma \Rightarrow \forall x F, \Delta} \forall R(*)$$

$$\frac{F[y/x], \Gamma \Rightarrow \Delta}{\exists x F, \Gamma \Rightarrow \Delta} \exists L(*)$$

$$\frac{\Gamma \Rightarrow F[t/x], \exists x F, \Delta}{\Gamma \Rightarrow \exists x F, \Delta} \exists R$$

(*): y not free in the conclusion of the rule

$\forall L$: $\forall x F$ can be dropped in the premise because Γ is a set

$\exists R$: $\exists x F$ can be dropped in the premise because Δ is a set

Natural Deduction rules

$$\frac{F[y/x]}{\forall x F} \forall I(*) \qquad \frac{\forall x F}{F[t/x]} \forall E$$
$$\frac{F[t/x]}{\exists x F} \exists I \qquad \frac{\exists x F \quad \begin{array}{c} [F[y/x]] \\ \vdots \\ H \end{array}}{H} \exists E(**)$$

(*): ($y = x$ or $y \notin fv(F)$) and
 y not free in an open assumption in the proof of $F[y/x]$

(**): ($y = x$ or $y \notin fv(F)$) and
 y not free in H or in an open assumption in the proof of the
second premise, except for $F[y/x]$

Hilbert System

Additional rule $\forall I$:

if F is provable then $\forall y F[y/x]$ is provable

provided y not free in the assumptions and ($y = x$ or $y \notin fv(F)$)

Additional axioms:

$$\forall x F \rightarrow F[t/x]$$

$$F[t/x] \rightarrow \exists x F$$

$$\forall x (G \rightarrow F) \rightarrow (G \rightarrow \forall y F[y/x]) \quad \text{if } y \notin fv(G), y = x \text{ or } y \notin fv(F)$$

$$\forall x (F \rightarrow G) \rightarrow (\exists y F[y/x] \rightarrow G) \quad \text{if } y \notin fv(G), y = x \text{ or } y \notin fv(F)$$