

Propositional Logic Basics

Syntax of propositional logic

Definition

An **atomic formula** (or **atom**) has the form A_i where $i = 1, 2, 3, \dots$

Formulas are defined inductively:

- ▶ \perp (“False”) and \top (“True”) are formulas
- ▶ All atomic formulas are formulas
- ▶ For all formulas F , $\neg F$ is a formula.
- ▶ For all formulas F und G , $(F \circ G)$ is a formula, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

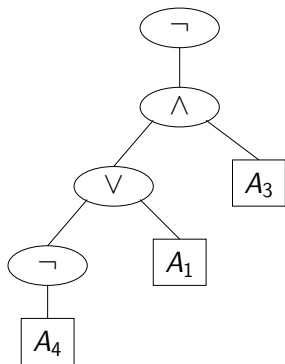
\neg	is called	negation
\wedge	is called	conjunction
\vee	is called	disjunction
\rightarrow	is called	implication
\leftrightarrow	is called	bi-implication

Syntax tree of a formula

Every formula can be represented by a syntax tree.

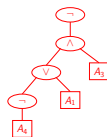
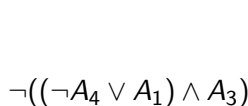
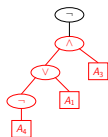
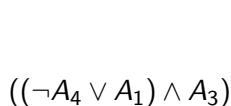
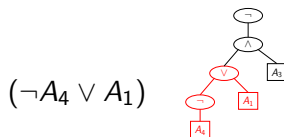
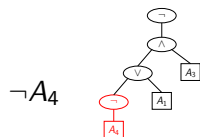
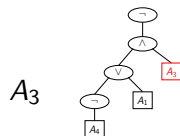
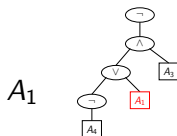
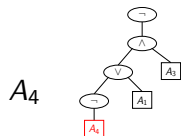
Example

$$F = \neg((\neg A_4 \vee A_1) \wedge A_3)$$



Subformulas

The **subformulas** of a formula are the formulas corresponding to the subtrees of its syntax tree.



Induction on formulas

Proof by induction on the structure of a formula:

In order to prove some property $\mathcal{P}(F)$ for all formulas F it suffices to prove the following:

- ▶ Base cases:
prove $\mathcal{P}(\perp)$, prove $\mathcal{P}(\top)$, and prove $\mathcal{P}(A_i)$ for all atoms A_i
- ▶ Induction step for \neg :
prove $\mathcal{P}(\neg F)$ under the induction hypothesis $\mathcal{P}(F)$
- ▶ Induction step for all $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$:
prove $\mathcal{P}(F \circ G)$ under the induction hypotheses $\mathcal{P}(F)$ and $\mathcal{P}(G)$

Operators that are merely abbreviations need not be considered!

Parentheses

Precedence of logical operators in decreasing order:

$$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$$

Operators with higher precedence bind more strongly.

Example

Instead of $(A \rightarrow ((B \wedge \neg(C \vee D)) \vee E))$

we can write $A \rightarrow B \wedge \neg(C \vee D) \vee E$.

Outermost parentheses can be dropped.

Semantics of propositional logic (I)

The elements of the set $\{0, 1\}$ are called **truth values**.
(You may call 0 “false” and 1 “true”)

An **assignment** is a function $\mathcal{A} : Atoms \rightarrow \{0, 1\}$
where *Atoms* is the set of all atoms.

We extend \mathcal{A} to a function $\hat{\mathcal{A}} : Formulas \rightarrow \{0, 1\}$

Semantics of propositional logic (II)

$$\hat{\mathcal{A}}(A_i) = \mathcal{A}(A_i)$$

$$\hat{\mathcal{A}}(\neg F) = \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\mathcal{A}}(F \wedge G) = \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 1 \text{ and } \hat{\mathcal{A}}(G) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\mathcal{A}}(F \vee G) = \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 1 \text{ or } \hat{\mathcal{A}}(G) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\mathcal{A}}(F \rightarrow G) = \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 0 \text{ or } \hat{\mathcal{A}}(G) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Instead of $\hat{\mathcal{A}}$ we simply write \mathcal{A}

Truth tables (I)

We can compute $\hat{\mathcal{A}}$ with the help of **truth tables**.

\neg	A
1	0
0	1

A	\vee	B
0	0	0
0	1	1
1	1	0
1	1	1

A	\wedge	B
0	0	0
0	0	1
1	0	0
1	1	1

A	\rightarrow	B
0	1	0
0	1	1
1	0	0
1	1	1

Abbreviations

$A, B, C,$
 $P, Q, R,$ or ... instead of $A_1, A_2, A_3 \dots$

$(F_1 \leftrightarrow F_2)$ abbreviates $((F_1 \wedge F_2) \vee (\neg F_1 \wedge \neg F_2))$

$(\bigvee_{i=1}^n F_i)$ abbreviates $(\dots ((F_1 \vee F_2) \vee F_3) \vee \dots \vee F_n)$

$(\bigwedge_{i=1}^n F_i)$ abbreviates $(\dots ((F_1 \wedge F_2) \wedge F_3) \wedge \dots \wedge F_n)$

Truth tables (II)

A	\leftrightarrow	B
0	1	0
0	0	1
1	0	0
1	1	1

Coincidence Lemma

Lemma

Let \mathcal{A}_1 and \mathcal{A}_2 be two assignments.

*If $\mathcal{A}_1(A_i) = \mathcal{A}_2(A_i)$ for all atoms A_i in some formula F ,
then $\mathcal{A}_1(F) = \mathcal{A}_2(F)$.*

Proof.

Exercise.



Models

If $\mathcal{A}(F) = 1$ then we write $\mathcal{A} \models F$
and say F is true under \mathcal{A}
or \mathcal{A} is a model of F

If $\mathcal{A}(F) = 0$ then we write $\mathcal{A} \not\models F$
and say F is false under \mathcal{A}
or \mathcal{A} is not a model of F

Validity and satisfiability

Definition (Validity)

A formula F is **valid** (or a **tautology**) if every assignment is a model of F .

We write $\models F$ if F is valid, and $\not\models F$ otherwise.

Definition (Satisfiability)

A formula F is **satisfiable** if it has at least one model; otherwise F is **unsatisfiable**.

A (finite or infinite!) set of formulas S is **satisfiable** if there is an assignment that is a model of every formula in S .

Exercise

	Valid	Satisfiable	Unsatisfiable
A			
$A \vee B$			
$A \vee \neg A$			
$A \wedge \neg A$			
$A \rightarrow \neg A$			
$A \rightarrow (B \rightarrow A)$			
$A \rightarrow (A \rightarrow B)$			
$A \leftrightarrow \neg A$			

Exercise

Which of the following statements are true?

	Y	C.ex.
If F is valid, then F is satisfiable		
If F is satisfiable, then $\neg F$ is satisfiable		
If F is valid, then $\neg F$ is unsatisfiable		
If F is unsatisfiable, then $\neg F$ is unsatisfiable		

Mirroring principle

all propositional formulas

valid formulas G	satisfiable but not valid formulas F $\neg F$	unsatisfiable formulas $\neg G$
------------------------------	--	---

Consequence

Definition

A formula G is a (semantic) consequence of a set of formulas M if every model \mathcal{A} of all $F \in M$ is also a model of G .

Then we write $M \models G$.

In a nutshell:

“Every model of M is a model of G .”

Example

$A \vee B, A \rightarrow B, B \wedge R \rightarrow \neg A, R \models (R \wedge \neg A) \wedge B$

Exercise

M	F	$M \models F?$
A	$A \vee B$	
A	$A \wedge B$	
A, B	$A \vee B$	
A, B	$A \wedge B$	
$A \wedge B$	A	
$A \vee B$	A	
$A, A \rightarrow B$	B	

Consequence, validity, satisfiability

Exercise

The following statements are equivalent:

1. $F_1, \dots, F_k \models G$
2. $\models (\bigwedge_{i=1}^k F_i) \rightarrow G$

Exercise

The following statements are equivalent:

1. $F \rightarrow G$ is valid.
2. $F \wedge \neg G$ is unsatisfiable.

Exercise

Let M be a set of formulas, and let F and G be formulas.
Which of the following statements hold?

	Y/N	C.ex.
If F satisfiable then $M \models F$.		
If F valid then $M \models F$.		
If $F \in M$ then $M \models F$.		
If $F \models G$ then $\neg F \models \neg G$.		

Notation

Warning: The symbol \models is overloaded:

$$\mathcal{A} \models F$$

$$\models F$$

$$M \models F$$

Convenient variations for set of formulas S :

$$\mathcal{A} \models S \text{ means that for all } F \in S, \mathcal{A} \models F$$

$$\models S \text{ means that for all } F \in S, \models F$$

$$M \models S \text{ means that for all } F \in S, M \models F$$