

# First-Order Logic

## The Classical Decision Problem

Validity/satisfiability of arbitrary first-order formulas is undecidable.

What about subclasses of formulas?

### Examples

$\forall x \exists y (P(x) \rightarrow P(y))$

Satisfiable? Resolution?

$\exists x \forall y (P(x) \rightarrow P(y))$

Satisfiable? Resolution?

# The $\exists^*\forall^*$ class

## Definition

The  $\exists^*\forall^*$  class is the class of closed formulas of the form

$$\exists x_1 \dots \exists x_m \forall y_1 \dots \forall y_n F$$

where  $F$  is quantifier-free and contains no function symbols of arity  $> 0$ .

This is also called the [Bernays-Schönfinkel class](#).

## Corollary

*Unsatisfiability is decidable for formulas in the  $\exists^*\forall^*$  class.*

What if a formula is not in the  $\exists^*\forall^*$  class?

Try to transform it into the  $\exists^*\forall^*$  class!

### Example

$$\forall y \exists x (P(x) \wedge Q(y))$$

Heuristic transformation procedure:

1. Put formula into NNF
2. Push all quantifiers into the formula as far as possible (“miniscoping”)
3. Pull out  $\exists$  first and  $\forall$  afterwards

# Miniscoping

Perform the following transformations bottom-up,  
as long as possible:

- ▶  $(\exists x F) \equiv F$  if  $x$  does not occur free in  $F$
- ▶  $\exists x (F \vee G) \equiv (\exists x F) \vee (\exists x G)$
- ▶  $\exists x (F \wedge G) \equiv (\exists x F) \wedge G$  if  $x$  is not free in  $G$
- ▶  $\exists x F$  where  $F$  is a conjunction,  
 $x$  occurs free in every conjunct,  
and the DNF of  $F$  is of the form  $F_1 \vee \dots \vee F_n$ ,  $n \geq 2$ :  
 $\exists x F \equiv \exists x (F_1 \vee \dots \vee F_n)$

Together with the dual transformations for  $\forall$

## Example

$$\exists x (P(x) \wedge \exists y (Q(y) \vee R(x)))$$

**Warning:** Complexity!

# The monadic class

## Definition

A formula is **monadic** if it contains only unary (monadic) predicate symbols and no function symbol of arity  $> 0$ .

## Examples

All men are mortal. Sokrates is a man. Sokrates is mortal.

# The monadic class is decidable

## Theorem

*Satisfiability of monadic formulas is decidable.*

**Proof** Put into NNF. Perform miniscoping.

The result has no nested quantifiers (**Exercise!**).

First pull out all  $\exists$ , then all  $\forall$ .

Existentially quantify free variables.

The result is in the  $\exists^*\forall^*$  class.

## Corollary

*Validity of monadic formulas is decidable.*

# The finite model property

## Definition

A formula  $F$  has the **finite model property** (for satisfiability) if  $F$  has a model iff  $F$  has a finite model.

## Theorem

*If a formula has the finite model property, satisfiability is decidable.*

## Theorem

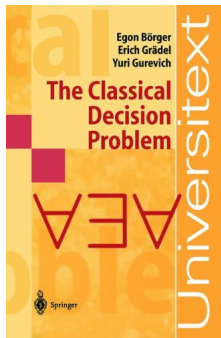
*Monadic formulas have the finite model property.*



# Classification by quantifier prefix of prenex form

There is a **complete** classification of decidable and undecidable classes of formulas based on

- ▶ the form of the quantifier prefix of the prenex form
- ▶ the arity of the predicate and function symbols allowed
- ▶ whether “=” is allowed or not.



# A complete classification

Only formulas without function symbols of arity  $> 0$ ,  
no restrictions on predicate symbols.

Satisfiability is decidable:

$\exists^* \forall^*$  (Bernays, Schönfinkel 1928)

$\exists^* \forall \exists^*$  (Ackermann 1928)

$\exists^* \forall^2 \exists^*$  (Gödel 1932)

Satisfiability is undecidable:

$\forall^3 \exists$  (Surányi 1959)

$\forall \exists \forall$  (Kahr, Moore, Wang 1962)

Why complete?

Famous mistake by Gödel:  $\exists^* \forall^2 \exists^*$  with “=” is **undecidable**  
(Goldfarb 1984)