

First-Order Logic Equality

Predicate logic with equality

Predicate logic
+
distinguished predicate symbol “=” of arity 2

Semantics: A structure \mathcal{A} of predicate logic with equality always maps the predicate symbol = to the identity relation:

$$\mathcal{A}(=) = \{(d, d) \mid d \in U_{\mathcal{A}}\}$$

Expressivity

Fact

A structure is model of $\exists x \forall y \ x=y$ iff its universe is a singleton.

Theorem

Every satisfiable formula of predicate logic has a countably infinite model.

Proof Let F be satisfiable.

We assume w.l.o.g. that $F = \forall x_1 \dots \forall x_n F^*$ and the variables occurring in F^* are exactly x_1, \dots, x_n .

(If necessary bring F into closed Skolem form).

We consider two cases:

$n = 0$. **Exercise.**

$n > 0$. Let $G = \forall x_1 \dots \forall x_n F^*[f(x_1)/x_1]$, where f is a function symbol that does not occur in F^* . G is satisfiable (**why?**) and $T(G)$ is countably infinite. It follows from the fundamental theorem that G has a countably infinite model.

Modelling equality

Let F be a formula of predicate logic with equality.

Let Eq be a predicate symbol that does not occur in F .

Let E_F be the conjunction of the following formulas:

$$\forall x \text{ Eq}(x, x)$$

$$\forall x \forall y (\text{Eq}(x, y) \rightarrow \text{Eq}(y, x))$$

$$\forall x \forall y \forall z ((\text{Eq}(x, y) \wedge \text{Eq}(y, z)) \rightarrow \text{Eq}(x, z))$$

For every function symbol f in F of arity n and every $1 \leq i \leq n$:

$$\forall x_1 \dots \forall x_n \forall y (\text{Eq}(x_i, y) \rightarrow \\ \text{Eq}(f(x_1, \dots, x_i, \dots, x_n), f(x_1, \dots, y, \dots, x_n)))$$

For every predicate symbol P in F of arity n and every $1 \leq i \leq n$:

$$\forall x_1 \dots \forall x_n \forall y (\text{Eq}(x_i, y) \rightarrow \\ (P(x_1, \dots, x_i, \dots, x_n) \leftrightarrow P(x_1, \dots, y, \dots, x_n)))$$

E_F expresses that Eq is a *congruence relation* on the symbols in F .

Quotient structure

Definition

Let \mathcal{A} be a structure and \sim an equivalence relation on $U_{\mathcal{A}}$ that is a congruence relation for all the predicate and function symbols defined by $I_{\mathcal{A}}$. The **quotient structure** \mathcal{A}/\sim is defined as follows:

- ▶ $U_{\mathcal{A}/\sim} = \{[u]_{\sim} \mid u \in U_{\mathcal{A}}\}$ where $[u]_{\sim} = \{v \in U_{\mathcal{A}} \mid u \sim v\}$
- ▶ For every function symbol f defined by $I_{\mathcal{A}}$:
 $f^{\mathcal{A}/\sim}([d_1]_{\sim}, \dots, [d_n]_{\sim}) = [f^{\mathcal{A}}(d_1, \dots, d_n)]_{\sim}$
- ▶ For every predicate symbol P defined by $I_{\mathcal{A}}$:
 $P^{\mathcal{A}/\sim}([d_1]_{\sim}, \dots, [d_n]_{\sim}) = P^{\mathcal{A}}(d_1, \dots, d_n)$
- ▶ For every variable x defined by $I_{\mathcal{A}}$: $x^{\mathcal{A}/\sim} = [x^{\mathcal{A}}]_{\sim}$

Lemma

$$\mathcal{A}/\sim(t) = [\mathcal{A}(t)]_{\sim}$$

Lemma

$$\mathcal{A}/\sim(F) = \mathcal{A}(F)$$

Theorem

The formulas F and $E_F \wedge F[Eq/=]$ are equisatisfiable.

Proof We show that if $E_F \wedge F[Eq/=]$ is sat., then F is satisfiable.

Assume $\mathcal{A} \models E_F \wedge F[Eq/=]$.

$\Rightarrow Eq^{\mathcal{A}}$ is an congruence relation.

Let $\mathcal{B} = \mathcal{A}/_{Eq^{\mathcal{A}}}$ (extended with $=$ interpreted as identity).

$\Rightarrow \mathcal{B} \models F[Eq/=]$

By construction $Eq^{\mathcal{B}}$ is identity.

$\Rightarrow \mathcal{B}(F[Eq/=]) = \mathcal{B}(F)$

$\Rightarrow \mathcal{B} \models F$