

Quantifier Elimination

Helpful lemmas

Let S be a set of sentences.

Lemma

$S \models F$ iff $S \models \forall F$

Lemma

*If $S \models F \leftrightarrow G$ then $S \models H[F] \leftrightarrow H[G]$,
i.e. one can replace a subterm F of H by G .*

Quantifier elimination

Definition

If $T \models F \leftrightarrow F'$ we say that F and F' are T -equivalent.

Definition

A theory T admits **quantifier elimination** if for every formula F there is a quantifier-free T -equivalent formula G such that $fv(G) \subseteq fv(F)$. We call G a **quantifier-free T -equivalent** of F .

Examples

In linear real arithmetic:

$$\exists x \exists y (3 * x + 5 * y = 7) \leftrightarrow ?$$
$$\forall y (x < y \wedge y < z) \leftrightarrow ?$$
$$\exists y (x < y \wedge y < z) \leftrightarrow ?$$

Quantifier elimination

A **quantifier-elimination procedure (QEP)** for a theory T and a set of formulas \mathcal{F} is a function that computes for every $F \in \mathcal{F}$ a quantifier-free T -equivalent.

Lemma

Let T be a theory such that

- ▶ *T has a QEP for all formulas and*
- ▶ *for all ground formulas G , $T \models G$ or $T \models \neg G$, and it is decidable which is the case.*

Then T is decidable and complete.

Simplifying quantifier elimination: one \exists

Fact

If T has a QEP for all $\exists x F$ where F is quantifier-free, then T has a QEP for all formulas.

Essence: It is sufficient to be able to eliminate a single \exists

Construction:

Given: a QEP $qe1$ for formulas of the form $\exists x F$ where F is quantifier-free

Define: a QEP for all formulas

Method: Eliminate quantifiers bottom-up by $qe1$, use $\forall \equiv \neg \exists \neg$

Simplifying quantifier elimination: $\exists x \wedge$ literals

Lemma

If T has a QEP for all $\exists x F$ where F is a conjunction of literals, all of which contain x , then T has a QEP for all $\exists x F$ where F is quantifier-free.

Construction:

Given: a QEP $qe1c$ for formulas of the form $\exists x (L_1 \wedge \dots \wedge L_n)$ where each L_i is a literal that contains x

Define: $qe1(\exists x F)$ where F is quantifier-free

Method: DNF; miniscoping; $qe1c$

This is the end of the generic part of quantifier elimination.
The rest is theory specific.

Eliminating “ \neg ”

Motivation: $\neg x < y \leftrightarrow y < x \vee y = x$ for linear orderings

Assume that there is a computable function *aneg* that maps every negated atom to a quantifier-free and negation-free *T*-equivalent formula.

Lemma

If T has a QEP for all $\exists x F$ where F is a conjunction of atoms, all of which contain x , then T has a QEP for all $\exists x F$ where F is quantifier-free.

Construction:

Given: a QEP *qe1ca* for formulas of the form $\exists x (A_1 \wedge \cdots \wedge A_n)$ where each atom A_i contains x

Define: *qe1*($\exists x F$) where F quantifier-free

Method: NNF; *aneg*; DNF; miniscoping; *qe1ca*

Quantifier Elimination
Dense Linear Orders
Without Endpoints

Dense Linear Orders Without Endpoints

$$\Sigma = \{<, =\}$$

Let **DLO** stand for “dense linear order without endpoints” and for the following set of axioms:

$$\forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$$

$$\forall x \neg(x < x)$$

$$\forall x \forall y (x < y \vee x = y \vee y < x)$$

$$\forall x \forall z (x < z \rightarrow \exists y (x < y \wedge y < z))$$

$$\forall x \exists y x < y$$

$$\forall x \exists y y < x$$

Models of DLO?

Theorem

All countable DLOs are isomorphic.

Quantifier elimination example

Example

$$DLO \models \exists y (x < y \wedge y < z) \leftrightarrow$$

Elimination of “ \neg ”

Elimination of negative literals (function *aneg*):

$$DLO \models \neg x = y \leftrightarrow x < y \vee y < x$$

$$DLO \models \neg x < y \leftrightarrow x = y \vee y < x$$

Quantifier elimination for conjunctions of atoms

QEP *qe1ca*($\exists x (A_1 \wedge \dots \wedge A_n)$ where x occurs in all A_i :

1. Eliminate “=”: Drop all A_i of the form $x = x$.

If some A_i is of the form $x = y$ (x and y different), eliminate $\exists x$:

$$\exists x (x = t \wedge F) \equiv F[t/x] \quad (x \text{ does not occur in } t)$$

Otherwise:

2. Eliminate $x < x$: return \perp

3. Separate atoms into lower and upper bounds for x and use

$$DLO \models \exists x \left(\bigwedge_{i=1}^m l_i < x \wedge \bigwedge_{j=1}^n x < u_j \right) \leftrightarrow \bigwedge_{i=1}^m \bigwedge_{j=1}^n l_i < u_j$$

Special case: $\bigwedge_{k=1}^0 F_k = \top$

Examples

$$\exists x (x < z \wedge y < x \wedge x < y') \leftrightarrow ?$$

$$\forall x (x < y) \leftrightarrow ?$$

$$\exists x \exists y \exists z (x < y \wedge y < z \wedge z < x) \leftrightarrow ?$$

Complexity

Quadratic blow-up with each elimination step

⇒ Eliminating all \exists from

$$\exists x_1 \dots \exists x_m F$$

where F has length n needs $O(\quad)$, assuming F is DNF.

Consequences

- ▶ $Th(DLO)$ has quantifier elimination
- ▶ $Th(DLO)$ is decidable and complete
- ▶ All models of DLO (for example $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$) are elementarily equivalent:
you cannot distinguish models of DLO by first-order formulas.

Quantifier Elimination

Linear real arithmetic

Linear real arithmetic

$$\mathcal{R}_+ = (\mathbb{R}, 0, 1, +, <, =), \quad R_+ = Th(\mathcal{R}_+)$$

For convenience we allow the following additional function symbols:

For every $c \in \mathbb{Q}$:

- ▶ c is a constant symbol
- ▶ $c \cdot$, multiplication with c , is a unary function symbol

A term in **normal form**: $c_1 \cdot x_1 + \dots + c_n \cdot x_n + c$

where $c_i \neq 0$, $x_i \neq x_j$ if $i \neq j$.

Every atom A is R_+ -equivalent to an atom $0 \bowtie t$ in **normal form (NF)** where $\bowtie \in \{<, =\}$ and t is in normal form.

An atom is **solved for x** if it is of the form $x < t$, $x = t$ or $t < x$ where x does not occur in t .

Any atom A in normal form that contains x can be transformed into an R_+ -equivalent atom solved for x .

Function $sol_x(A)$ solves A for x .

Elimination of “ \neg ”

Elimination of negative literals (function *aneg*):

$$R_+ \models \neg x = y \leftrightarrow x < y \vee y < x$$

$$R_+ \models \neg x < y \leftrightarrow x = y \vee y < x$$

Fourier-Motzkin Elimination

QEP $qe1ca(\exists x (A_1 \wedge \dots \wedge A_n)$, all A_i in NF and contain x :

1. Let $S = \{sol_x(A_1), \dots, sol_x(A_n)\}$

2. Eliminate “=”:

If $(x = t) \in S$ for some t , eliminate $\exists x$:

$$\exists x (x = t \wedge F) \equiv F[t/x] \quad (x \text{ does not occur in } t)$$

Otherwise return

$$\bigwedge_{(l < x) \in S} \bigwedge_{(x < u) \in S} l < u$$

Special case: empty \bigwedge is \top

All returned formulas are implicitly put into NF.

Examples

$$\exists x \exists y (3x + 5y < 7 \wedge 2z - 3y < 2) \leftrightarrow ?$$

$$\exists x \forall y (3y \leq x \vee x \leq 2y) \leftrightarrow ?$$

Can DNF be avoided?

Ferrante and Rackoff's theorem

Theorem

Let F be quantifier-free and negation-free and assume all atoms that contain x are solved for x . Let S_x be the set of atoms that contain x . Let $L = \{l \mid (l < x) \in S_x\}$, $U = \{u \mid (x < u) \in S_x\}$, $E = \{t \mid (x = t) \in S_x\}$. Then

$$R_+ \models \exists x F \leftrightarrow F[-\infty/x] \vee F[\infty/x] \vee \bigvee_{t \in E} F[t/x] \vee \bigvee_{l \in L} \bigvee_{u \in U} F[0.5(l+u)/x]$$

(note: empty \bigvee is \perp) where $F[-\infty/x]$ ($F[\infty/x]$) is the following transformation of all solved atoms in F :

$$x < t \mapsto \top (\perp)$$

$$t < x \mapsto \perp (\top)$$

$$x = t \mapsto \perp (\perp)$$

Examples

$$\exists x (y < x \wedge x < z) \leftrightarrow ?$$

$$\exists x x < y \leftrightarrow ?$$

Ferrante and Rackoff's procedure

Define $qe1(\exists x F)$:

1. Put F into NNF, eliminate all negations, put all atoms into normal form, solve those atoms for x that contain x .
2. Apply Ferrante and Rackoff's theorem.

Theorem

Eliminating all quantifiers with Ferrante and Rackoff's procedure from a formula of size n takes space $O(2^{cn})$ and time $O(2^{2^{dn}})$.

Quantifier Elimination Linear Integer Arithmetic

See [Harrison] or [Enderton] under “Presburger”