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Submission of homework: Before tutorial on 13.06.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 6.1. **[Unused Bound Variables]**
For this exercise, we first define an alternative way to evaluate a formula in a structure, based on arithmetic and set operations.

$$
\begin{aligned}
\mathcal{A}(\neg F) &= 1 - \mathcal{A}(F) \\
\mathcal{A}(F \vee G) &= \max(\mathcal{A}(F), \mathcal{A}(G)) \\
\mathcal{A}(F \wedge G) &= \min(\mathcal{A}(F), \mathcal{A}(G)) \\
\mathcal{A}(\exists x\ F) &= \max\{\mathcal{A}[d/x](F) \mid d \in U_{\mathcal{A}}\} \\
\mathcal{A}(\forall x\ F) &= \min\{\mathcal{A}[d/x](F) \mid d \in U_{\mathcal{A}}\}
\end{aligned}
$$

The evaluation for predicates and terms remain unchanged.

Equipped with this definition, prove the equivalence $\exists x F \equiv F$ where x does not occur in F.

Hint: Adapt the coincidence lemma for propositional logic (exercise 1.1) to predicate logic.

Exercise 6.2. **[Substitution Lemma]**
Consider the following statement: "If $F \equiv F'$, then $F[t/x] \equiv F'[t/x]$." Proof or counterexample.

Exercise 6.3. **[Skolem Form]**
Convert the following formula into – in order – a rectified formula, RPF and Skolem form.

$$P(x) \wedge \forall x\ (Q(x) \wedge \forall x \exists y\ P(f(x, y)))$$

Exercise 6.4. **[Herbrand Models]**
Given the formula

$$F = \forall x \forall y (P(f(x), g(y)) \wedge \neg P(g(x), f(y)))$$

1. Specify a Herbrand model for F.
2. Specify a Herbrand structure suitable for F, which is not a model of F.

Homework 6.1. **[Skolem Form]** (6 points)
Convert the following formulas into – in order – a rectified formula, RPF and Skolem form.

1. $\forall x \exists y \forall z \exists w (\neg P(a, w) \vee Q(f(x), y))$
2. $\forall z \exists y (P(x, g(y), z) \vee \neg \forall x \, Q(x))$

Homework 6.2. **[Invalid Herbrand Models]** (8 points)
Recall the fundamental theorem from the lecture: "Let F be a closed formula in Skolem form. Then F is satisfiable iff it has a Herbrand model".

Explain "what goes wrong" if the precondition is violated: when F is not closed or not in Skolem form. Describe both cases.

Homework 6.3. **[Proof of the Fundamental Theorem]** (6 points)
Recall the proof of the fundamental theorem from the lecture. Give the omitted proof for the base case (slide 6, $\mathcal{A}(G) = \mathcal{T}(G)$).