

LOGICS EXERCISE

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EXERCISE SHEET 4

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Submission of homework: Before tutorial on 30.05.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 4.1. [Natural Deduction]

Prove the following formulas by natural deduction:

1. $(F \wedge G) \wedge H \rightarrow F \wedge (G \wedge H)$
2. $(F \vee G) \vee H \rightarrow F \vee (G \vee H)$
3. $\neg(F \wedge G) \rightarrow (\neg F \vee \neg G)$

Solution:

Exercise 4.2. [Induction Proofs]

Show that the following schema has a proof in natural deduction for all $n \geq 1$:

$$P_n = ((A_1 \wedge (A_2 \wedge (\dots \wedge A_n) \dots)) \rightarrow B) \rightarrow (A_1 \rightarrow (A_2 \rightarrow (\dots (A_n \rightarrow B) \dots))))$$

Solution:

- Base case: $n = 1$

Formula: $P_1 = (A_1 \rightarrow B) \rightarrow (A_1 \rightarrow B)$

Proof: trivial through $\rightarrow I$

Exercise 4.2

- Induction step

Subproof

$$\begin{array}{c}
 [A_1 \wedge \dots \wedge A_{n+1} \rightarrow B]^1 \\
 \vdots \\
 A_1 \wedge \dots \wedge A_{n+1} \rightarrow B \\
 \hline
 B
 \end{array}
 \qquad
 \begin{array}{c}
 [A_1]^2 \quad [A_2 \wedge \dots \wedge A_{n+1}]^3 \\
 \vdots \quad \vdots \\
 A_1 \quad A_2 \wedge \dots \wedge A_{n+1} \quad \wedge \\
 \hline
 A_1 \wedge \dots \wedge A_{n+1} \quad \rightarrow E
 \end{array}$$

$$\begin{array}{c}
 [IH]^* \\
 \hline
 (A_2 \wedge \dots \wedge A_{n+1} \rightarrow B) \rightarrow (A_2 \rightarrow \dots \rightarrow A_{n+1} \rightarrow B) \\
 \hline
 A_2 \rightarrow \dots \rightarrow A_{n+1} \rightarrow B \quad \rightarrow I (2) \\
 \hline
 A_2 \rightarrow \dots \rightarrow A_{n+1} \rightarrow B \quad (1) \\
 \hline
 (A_1 \wedge \dots \wedge A_{n+1} \rightarrow B) \rightarrow (A_1 \rightarrow \dots \rightarrow A_{n+1} \rightarrow B) \quad \rightarrow I
 \end{array}$$

* note: we use the induction hypothesis with a.f. $\{A_2, \dots, A_{n+1}\}$

Exercise 4.3. [Classical Reasoning]

We replace rule \perp of the calculus of natural deduction by either one of the following rules:

- $\frac{}{F \vee \neg F}$ (law of excluded middle)
- $\frac{\neg\neg F}{F}$ (double negation elimination)

Additionally, we add the rule $\frac{\perp}{F}$ ($\perp E$). Show that the calculus of natural deduction remains complete in both cases.

Solution:

We want to show that the \perp rule can be derived from either of the two other alternatives. Thus we assume that there is a proof of the form

$$\begin{array}{c} \neg F \\ \vdots \\ \perp \end{array}$$

and we need to show that we then can also prove F .

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$$\frac{\frac{}{F \vee \neg F} \text{ (law of excluded middle)} \quad \frac{[\neg F]^1 \quad \frac{\perp}{F} \perp E}{\vee E} (1)}{F}$$

•

$$\frac{\frac{[\neg F]^1 \quad \frac{\perp}{\neg\neg F} \neg I (1)}{\frac{\perp}{F} \text{ (double negation elimination)}}$$

Homework 4.1. [Natural Deduction]

(10 points)

Prove the following formulas by natural deduction (as specified in the lecture):

1. $((A \rightarrow B) \rightarrow A) \rightarrow A$
2. $(A \vee B) \rightarrow \neg(\neg A \wedge \neg B)$
3. $(\neg G \rightarrow F) \rightarrow (\neg F \rightarrow G)$

Homework 4.2. [Substitution]

(10 points)

Assume that there are proofs for $\vdash_N G \rightarrow G'$ and $\vdash_N G' \rightarrow G$. Construct the proof for $\vdash_N F[G/A] \rightarrow F[G'/A]$.