

LOGICS EXERCISE

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EXERCISE SHEET 5

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**Submission of homework:** Before tutorial on **Wednesday, 07.06.2017**. You have to do the homework yourself; no teamwork allowed.

**Exercise 5.1. [From Natural Deduction to Hilbert Calculus]**

Prove the following formula to a linear proof in Hilbert calculus:

$$(F \wedge G) \rightarrow (G \wedge F)$$

*Hint:* Use the deduction theorem.

**Solution:**

We first apply the deduction theorem. It remains to construct a proof of  $F \wedge G \vdash_H G \wedge F$ .

- |   |          |
|---|----------|
| 1. $F \wedge G \rightarrow G$               | A4       |
| 2. $F \wedge G$                             | $\Gamma$ |
| 3. $F \wedge G \rightarrow F$               | A3       |
| 4. $G \rightarrow F \rightarrow G \wedge F$ | A5       |
| 5. $G$                                      | 1, 2     |
| 6. $F \rightarrow G \wedge F$               | 4, 5     |
| 7. $F$                                      | 3, 2     |
| 8. $G \wedge F$                             | 6, 7     |

**Exercise 5.2. [Equivalence]**

Let  $F$  and  $G$  be arbitrary formulas. (In particular, they may contain free occurrences of  $x$ .) Which of the following equivalences hold? Proof or counterexample!

1.  $\forall x(F \wedge G) \equiv \forall xF \wedge \forall xG$
2.  $\exists x(F \wedge G) \equiv \exists xF \wedge \exists xG$

**Solution:**

1. Holds. Assume  $\mathcal{A} \models \forall x(F \wedge G)$ ,  
 $\iff$  for all  $d \in U_{\mathcal{A}}$ , we have  $\mathcal{A}[d/x] \models F$  and  $\mathcal{A}[d/x] \models G$ ,  
 $\iff$  for all  $d_1 \in U_{\mathcal{A}}$ , we have  $\mathcal{A}[d_1/x] \models F$  and for all  $d_2 \in U_{\mathcal{A}}$ , we have  $\mathcal{A}[d_2/x] \models G$   
 $\iff \mathcal{A} \models \forall xF \wedge \forall xG$
2. Does not hold. Let  $F = P(x)$  and  $G = Q(x)$ ,  $U_{\mathcal{A}} = \{0, 1\}$ ,  $P^{\mathcal{A}} = \{0\}$ , and  $Q^{\mathcal{A}} = \{1\}$ . Clearly,  $\mathcal{A} \models \exists xF \wedge \exists xG$  but  $\mathcal{A} \not\models \exists x(F \wedge G)$

**Exercise 5.3. [Infinite Models]**

Consider predicate logic with equality. We use infix notation for equality, and abbreviate  $\neg(s = t)$  by  $s \neq t$ . Moreover, we call a structure finite iff its universe is finite.

1. Specify a finite model for the formula  $\forall x (c \neq f(x) \wedge x \neq f(x))$ .
2. Specify a model for the formula  $\forall x \forall y (c \neq f(x) \wedge (f(x) = f(y) \longrightarrow x = y))$ .
3. Show that the above formula has no finite model.

**Solution:**

1.  $U^{\mathcal{A}} = \{0, 1, 2\} \subset \mathbb{N}$  and  $c^{\mathcal{A}} = 0$  and  $f^{\mathcal{A}}(0) = 1 \mid f^{\mathcal{A}}(n + 1) = 2 - n$
2.  $U^{\mathcal{A}} = \mathbb{N}$  and  $c^{\mathcal{A}} = 0$  and  $f^{\mathcal{A}}(n) = n + 1$
3. Assume a model  $\mathcal{A}$ . First note that the properties transfer to the semantic level, i.e., we have for all  $x, y \in U_{\mathcal{A}}$ :

$$c^{\mathcal{A}} \neq f^{\mathcal{A}}(x) \tag{1}$$

$$f^{\mathcal{A}}(x) = f^{\mathcal{A}}(y) \implies x = y \tag{2}$$

Now, we are in a position to show that  $U_{\mathcal{A}}$  is infinite. For this, we define  $x_i = (f^{\mathcal{A}})^i(c^{\mathcal{A}})$ , i.e.  $i$  times  $f^{\mathcal{A}}$  applied to  $c^{\mathcal{A}}$ . Clearly, we have  $x_i \in U_{\mathcal{A}}$  for all  $i$ . We now show that  $i < j$  implies  $x_i \neq x_j$ , immediately yielding infinity of  $U_{\mathcal{A}}$ . We do induction on  $i$ . For 0, we have  $x_0 = c^{\mathcal{A}} \neq f^{\mathcal{A}}(\dots) = x_j$  (by (1)). For  $i + 1$ , the induction hypothesis gives us  $x_i \neq x_j$ , which implies  $x_{i+1} \neq x_{j+1}$  (by (2)). qed.

**Homework 5.1.** [A Smaller Set of Axioms for Hilbert Calculus] (6 points)

In the lecture, Hilbert calculus for propositional logic was introduced by means of nine axioms. However, the following three axioms are already sufficient:

$$\mathbf{A1} \quad F \rightarrow (G \rightarrow F)$$

$$\mathbf{A2} \quad (F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$$

$$\mathbf{A10} \quad (\neg F \rightarrow \neg G) \rightarrow (G \rightarrow F)$$

Derive the following axioms from the axioms above with help of  $\rightarrow_E$ :

$$1. \quad \neg(F \rightarrow F) \rightarrow G$$

$$2. \quad \neg\neg F \rightarrow F$$

**Homework 5.2.** [Predicate Logic] (6 points)

1. Specify a satisfiable formula  $F$  such that for all models  $\mathcal{A}$  of  $F$ , we have  $|U_{\mathcal{A}}| \geq 3$ .

2. Can you also specify a satisfiable formula  $F$  such that for all models  $\mathcal{A}$  of  $F$ , we have  $|U_{\mathcal{A}}| \leq 3$ ?

**Homework 5.3.** [Orders] (8 points)

A reflexive and transitive relation is called *preorder*. In predicate logic, preorders can be characterized by the formula

$$F \equiv \forall x \forall y \forall z (P(x, x) \wedge (P(x, y) \wedge P(y, z) \rightarrow P(x, z)))$$

Which of the following structures are models of  $F$ ? Give an informal proof in the positive case and a counterexample for the negative case!

$$1. \quad U^{\mathcal{A}} = \mathbb{N} \text{ and } P^{\mathcal{A}} = \{(m, n) \mid m < n\}$$

$$2. \quad U^{\mathcal{A}} = \mathbb{Z} \times \mathbb{Z} \text{ and } P^{\mathcal{A}} = \{((x, y), (a, b)) \mid a - b \leq x - y\}$$

$$3. \quad U^{\mathcal{A}} = \mathbb{R} \text{ and } P^{\mathcal{A}} = \{(m, n) \mid m = n\}$$

4. Let  $Q(x, y)$  be specified as follows:  $\forall x \forall y (P(x, y) \leftrightarrow Q(y, x))$ . Is  $Q$  a preorder?

Specify the notion of *partial orders*, that is, preorders that additionally satisfy antisymmetry.