Submission of homework: Before tutorial on 13.06.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 6.1. **[Unused Bound Variables]**
For this exercise, we first define an alternative way to evaluate a formula in a structure, based on arithmetic and set operations.

$$
\begin{aligned}
\mathcal{A}(\neg F) &= 1 - \mathcal{A}(F) \\
\mathcal{A}(F \vee G) &= \max(\mathcal{A}(F), \mathcal{A}(G)) \\
\mathcal{A}(F \wedge G) &= \min(\mathcal{A}(F), \mathcal{A}(G)) \\
\mathcal{A}(\exists x\ F) &= \max\{\mathcal{A}[d/x](F) \mid d \in U_{\mathcal{A}}\} \\
\mathcal{A}(\forall x\ F) &= \min\{\mathcal{A}[d/x](F) \mid d \in U_{\mathcal{A}}\}
\end{aligned}
$$

The evaluation for predicates and terms remain unchanged.

Equipped with this definition, prove the equivalence $\exists x F \equiv F$ where x does not occur in F.

Hint: Adapt the coincidence lemma for propositional logic (exercise 1.1) to predicate logic.

Solution:
Reminder: The coincidence lemma for propositional logic states that, assuming that for all atomic formulas A_i in F, $\mathcal{A}(A_i) = \mathcal{A}'(A_i)$, the statements $\mathcal{A} \models F$ and $\mathcal{A}' \models F$ are equivalent.

We formulate the coincidence lemma for predicate logic. Let $FV(t)$ denote the variables occurring in a term and $FV(f)$ the free variables occurring in a formula.

Let t be a term. Assume that two structures $\mathcal{A}$ and $\mathcal{A}'$ have identical interpretations for function and predicate symbols. Additionally, assume that for all variables $x \in FV(t)$, $x^{\mathcal{A}} = x^{\mathcal{A}'}$. Then, $\mathcal{A}(t) = \mathcal{A}'(t)$.

Proof: Simple, via induction over t.

Similarly, this can be applied to a formula F, i.e., $\mathcal{A}(F) = \mathcal{A}'(F)$ if $\mathcal{A}$ and $\mathcal{A}'$ coincide on $FV(F)$.

Finally, we can prove the equivalence stated in the exercise. Let $\mathcal{A}$ be an arbitrary structure suitable for F. We know that $FV(\exists x F) = FV(F)$, because x does not occur in F. Hence, $\mathcal{A}$ is also suitable for $\exists x F$.

$$
\begin{aligned}
\mathcal{A} \models \exists x F \text{ iff } & \max\{\mathcal{A}[d/x](F) \mid d \in U_{\mathcal{A}}\} = 1 \\
\text{iff } & \max\{\mathcal{A}(F) \mid d \in U_{\mathcal{A}}\} = 1 \quad \text{(by coincidence: } x \notin FV(F)) \\
\text{iff } & \max\{\mathcal{A}(F)\} = 1 \\
\text{iff } & \mathcal{A} \models F
\end{aligned}
$$

Exercise 6.2. **[Substitution Lemma]**
Consider the following statement: "If $F \equiv F'$, then $F[t/x] \equiv F'[t/x]$." Proof or counterexample.

Solution:
We can easily construct a counterexample because our definition of substitution does not take bound variables into account properly. Hence, we can "capture" bound variables during a substitution.

Counterexample:

$$\begin{aligned} F &= \forall z P(x) \\ F' &= \forall y P(x) \\ F[y/x] &= \forall z P(y) \\ F'[y/x] &= \forall y P(y) \end{aligned}$$

Clearly, $\forall x P(x) \not\equiv \forall y P(x)$. To fix this, we can forbid substitution of terms with free variables or rename bound variables during substitution.

Exercise 6.3. **[Skolem Form]**
Convert the following formula into – in order – a rectified formula, RPF and Skolem form.

$$P(x) \wedge \forall x\ (Q(x) \wedge \forall x \exists y\ P(f(x,y)))$$

Solution:

$$\begin{aligned} & P(x) \wedge \forall x (Q(x) \wedge \forall x \exists y P(f(x,y))) && \\ \rightsquigarrow & P(x) \wedge \forall x_1 (Q(x_1) \wedge \forall x_2 \exists y P(f(x_2,y))) && \text{rectified} \\ \rightsquigarrow & \exists x P(x) \wedge \forall x_1 (Q(x_1) \wedge \forall x_2 \exists y P(f(x_2,y))) && \text{rectified and closed} \\ \rightsquigarrow & \exists x \forall x_1 \forall x_2 \exists y (P(x) \wedge (Q(x_1) \wedge P(f(x_2,y)))) && \text{RPF} \\ \rightsquigarrow & \forall x_1 \forall x_2 (P(g) \wedge (Q(x_1) \wedge P(f(x_2, h(x_1,x_2))))) && \text{Skolem form} \end{aligned}$$

Exercise 6.4. **[Herbrand Models]**
Given the formula

$$F = \forall x \forall y (P(f(x), g(y)) \wedge \neg P(g(x), f(y)))$$

1. Specify a Herbrand model for F.
2. Specify a Herbrand structure suitable for F, which is not a model of F.

Solution:
We define $U_{\mathcal{A}} = D(F)$, i.e., the Herbrand universe for F. Note that we have a constant $a \in D(F)$. We define $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ to be the Herbrand-interpretations.

1. We define $P^{\mathcal{A}} = \{(f(t_1), g(t_2)) \mid t_1, t_2 \in D(F)\}$.
2. We define $P^{\mathcal{A}} = \{(g(t_1), f(t_2)) \mid t_1, t_2 \in D(F)\}$.

Homework 6.1. [Skolem Form] (6 points)
Convert the following formulas into – in order – a rectified formula, RPF and Skolem form.

1. $\forall x \exists y \forall z \exists w(\neg P(a, w) \vee Q(f(x), y))$
2. $\forall z \exists y(P(x, g(y), z) \vee \neg \forall x\, Q(x))$

Homework 6.2. [Invalid Herbrand Models] (8 points)
Recall the fundamental theorem from the lecture: "Let F be a closed formula in Skolem form. Then F is satisfiable iff it has a Herbrand model".

Explain "what goes wrong" if the precondition is violated: when F is not closed or not in Skolem form. Describe both cases.

Homework 6.3. [Proof of the Fundamental Theorem] (6 points)
Recall the proof of the fundamental theorem from the lecture. Give the omitted proof for the base case (slide 6, $\mathcal{A}(G) = \mathcal{T}(G)$).