

LOGICS EXERCISE

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EXERCISE SHEET 11

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Submission of homework: Before tutorial on 18.07.2017. You have to do the homework yourself; no teamwork allowed.

Exercise 11.1. [Decidable Theories]

Let S be a set of sentences (i.e. closed formulas) such that S is closed under consequence: if $S \models F$ and F is closed, then $F \in S$. Additionally, assume that S is finitely axiomatizable and complete, i.e. $F \in S$ or $\neg F \in S$ for any sentence F .

1. Give a procedure for deciding whether $S \models F$ for a sentence F .
2. Can you obtain a similar result when the assumption is that the axiom system is only *recursively enumerable*?

Solution:

1. Let M be the set of axioms. Run resolution on $M \wedge F$ and $M \wedge \neg F$ in parallel. If $F \notin S$, then $M \wedge F \vdash \square$ and the first resolution terminates. If $F \in S$, then $M \wedge \neg F \vdash \square$ and the second resolution terminates.
2. Yes, by compactness. Enumerate all finite subsets of the axiom set and run resolutions in parallel.

Exercise 11.2. [QE for DLO]

Use the quantifier-elimination procedure for DLOs to check whether the following formula is a member of $Th(DLO)$:

$$\exists x \forall y \exists z ((x < y \vee z < x) \wedge y < z)$$

Solution:

$$\begin{aligned} & \exists x \forall y \exists z ((x < y \vee z < x) \wedge y < z) \\ \iff & \exists x \forall y (\exists z (x < y \wedge y < z) \vee \exists z (z < x \wedge y < z)) \\ \iff & \exists x \forall y ((x < y \wedge \exists z (y < z)) \vee \exists z (z < x \wedge y < z)) \\ \iff & \exists x \forall y ((x < y \wedge \top) \vee \exists z (z < x \wedge y < z)) \\ \iff & \exists x \forall y (x < y \vee y < x) \\ \iff & \exists x \neg \exists y \neg (x < y \vee y < x) \\ \iff & \exists x \neg \exists y ((y < x \vee x = y) \wedge (x < y \vee x = y)) \\ \iff & \exists x \neg \exists y ((y < x \wedge x < y) \vee (y < x \wedge x = y) \vee (x = y \wedge x < y) \vee (x = y)) \\ \iff & \exists x \neg \exists y (\perp \vee \perp \vee \perp \vee (x = y)) \\ \iff & \exists x \neg \top \iff \perp \end{aligned}$$

Exercise 11.3. [Consequence]

Show that Cn is a closure operator, i.e. Cn fulfills the following properties:

- $S \subseteq Cn(S)$
- if $S \subseteq S'$ then $Cn(S) \subseteq Cn(S')$
- $Cn(Cn(S)) = Cn(S)$

Solution:

In the following, suppose S, S' are sets of Σ -sentences and F is Σ -sentence.

$$F \in S \implies S \models F \implies Cn(S) \models F$$

$$F \in Cn(S) \implies S \models F \implies S' \models F \implies F \in Cn(S')$$

From the above two: $Cn(S) \subseteq Cn(Cn(S))$

$$F \in Cn(Cn(S)) \implies Cn(S) \models F \implies^{(*)} S \models F \implies F \in Cn(S)$$

We have (*) because $\mathcal{A} \models Cn(S)$ iff $\mathcal{A} \models S$ by definition of Cn .

Exercise 11.4. [Axiomatizations and Compactness]

The compactness theorem for first-order logic: *If every finite subset of a (countable) set M of formulas has a model, then M as a whole has a model.*

Using compactness, show that if a theory is finitely axiomatizable, any countable axiomatization of it has a finite subset that axiomatizes the same theory. In other words, if $Cn(\Gamma) = Cn(\Delta)$ with Γ countable and Δ finite, then there is a finite $\Gamma' \subseteq \Gamma$ with $Cn(\Gamma') = Cn(\Gamma)$.

Solution:

Claim: We can construct a finite subset $\Gamma' \subseteq \Gamma$ that axiomatizes $Cn(\Delta)$. In particular, $\Gamma' \vdash \Delta$ must hold. This is equivalent to $\Gamma', \neg\Delta \vdash \perp$.

We also know that $\Gamma, \neg\Delta \vdash \perp$, because Γ axiomatizes $Cn(\Delta)$. Hence, the infinite set of formulas $\Gamma \cup \{\neg\Delta\}$ is unsatisfiable. By compactness, there must be a finite subset that is unsatisfiable.

We can find this subset by enumerating all finite subsets $\Gamma' \subseteq \Gamma$ and running resolution on $\Gamma', \neg\Delta$.

Homework 11.1. [QE for DLO] (8 points)

Use the quantifier-elimination procedure for DLOs to check whether the following formulas are member of $Th(DLO)$:

1. $\forall x \forall y \forall z (x < y \rightarrow (y < z \rightarrow x < z))$
2. $\exists x \exists y \forall z ((z < x \rightarrow z \leq y) \wedge (z > y \rightarrow z \geq x))$

Hint: Assume that \rightarrow is defined in terms of \vee and \neg ; \leq in terms of $=$, $<$ and \vee (and similar for $>$ and \geq).

Homework 11.2. [Refining Fourier-Motzkin] (6 points)

Show how Fourier-Motzkin elimination can be extended to directly handle constraints of the form $x \leq y$ instead of rewriting them to $x < y \vee x = y$ first.

Homework 11.3. [Difference Logic] (6 points)

We consider a fragment of linear arithmetic, in which atomic formulas only take the form $x - y \leq c$ for variables x and y , and $c \in \mathbb{Q}$.

For a finite set S of such difference constraints, we can define a corresponding inequality graph $G(V, E)$, where V is the set of variables of S , and E consists of all the edges (x, y) with weight c for all constraints $x - y \leq c$ of S . Show that the conjunction of all constraints from S is satisfiable iff G does not contain a negative cycle.

How can you use this theorem to obtain a procedure for deciding whether a formula is a member of this fragment?