| Logics Exercise | | |
|---|---|---|
| | TU München<br>Institut für Informatik | Prof. Tobias Nipkow<br>Lars Hupel |
| SS 2017 | Exercise Sheet 12 | 18.07.2017 |

**Submission of homework:** Before tutorial on 25.07.2017. You have to do the homework yourself; no teamwork allowed.

**Exercise 12.1.** **[Fourier–Motzkin Elimination]**

1. $\exists x \exists y (2 \cdot x + 3 \cdot y = 7 \land x < y \land 0 < x)$
2. $\exists x \exists y (3 \cdot x + 3 \cdot y < 8 \land 8 < 3 \cdot x + 2 \cdot y)$

**Solution:**

$$
\begin{aligned}
&\exists x \exists y (2 \cdot x + 3 \cdot y = 7 \land x < y \land 0 < x) \\
\leftrightarrow\, &\exists x \left( \exists y \left( y = \frac{7}{3} - \frac{2}{3} \cdot x \land x < y \right) \land 0 < x \right) \\
\leftrightarrow\, &\exists x \left( x < \frac{7}{3} - \frac{2}{3} \cdot x \land 0 < x \right) \\
\leftrightarrow\, &\exists x \left( x < \frac{7}{5} \land 0 < x \right) \\
\leftrightarrow\, &0 < \frac{7}{5} \leftrightarrow \top
\end{aligned}
$$

$$
\begin{aligned}
&\exists x \exists y (3 \cdot x + 3 \cdot y < 8 \land 8 < 3 \cdot x + 2 \cdot y) \\
\leftrightarrow\, &\exists x \exists y \left( y < \frac{8}{3} - x \land 4 - \frac{3}{2} \cdot x < y \right) \\
\leftrightarrow\, &\exists x \left( 4 - \frac{3}{2} \cdot x < \frac{8}{3} - x \right) \\
\leftrightarrow\, &\exists x \left( \frac{8}{3} < x \right) \leftrightarrow \top
\end{aligned}
$$

**Exercise 12.2.** **[Ferrante–Rackoff Elimination]**

$$\exists x(\exists y(x = 2 \cdot y) \to (2 \cdot x \geq 0 \lor 3 \cdot x < 2))$$

**Solution:**

$$\begin{aligned}
&\exists x(\exists y(x = 2 \cdot y) \to (2 \cdot x \geq 0 \lor 3 \cdot x < 2)) \\
\leftrightarrow\, &\exists x(\top \to (2 \cdot x \geq 0 \lor 3 \cdot x < 2)) \\
\leftrightarrow\, &\exists x(2 \cdot x \geq 0 \lor 3 \cdot x < 2) \\
\leftrightarrow\, &\exists x\left(0 < x \lor x = 0 \lor x < \frac{2}{3}\right) \\
\leftrightarrow\, &\left(\top \lor \top \lor \left(0 < 0 \lor 0 = 0 \lor 0 < \frac{2}{3}\right) \lor \cdots\right) \\
\leftrightarrow\, &\top
\end{aligned}$$

**Exercise 12.3.** **[Presburger Arithmetic]**

1. $\forall y(3 < x + 2 \cdot y \lor 2 \cdot x + y < 3)$
2. $\forall x(\exists y(x = 2 \cdot y) \to (2 \cdot x \geq 0 \lor 3 \cdot x < 2))$

**Solution:**

$$\begin{aligned}
&\forall y(3 < x + 2 \cdot y \lor 2 \cdot x + y < 3) \\
\leftrightarrow\, &\neg\exists y \neg(3 < x + 2 \cdot y \lor 2 \cdot x + y < 3) \\
\leftrightarrow\, &\neg\exists y(3 \geq x + 2 \cdot y \land 2 \cdot x + y \geq 3) \\
\leftrightarrow\, &\neg\exists y(2 \cdot y \leq 3 - x \land 3 - 2 \cdot x \leq y) \\
\leftrightarrow\, &\neg\exists y(2 \cdot y \leq 3 - x \land 6 - 4 \cdot x \leq 2 \cdot y) \\
\leftrightarrow\, &\neg\exists z(z \leq 3 - x \land 6 - 4 \cdot x \leq z \land 2 \mid z) \\
\leftrightarrow\, &\neg((6 - 4 \cdot x \leq 3 - x \land 2 \mid 6 - 4 \cdot x) \lor (7 - 4 \cdot x \leq 3 - x \land 2 \mid 7 - 4 \cdot x))
\end{aligned}$$

$$\begin{aligned}
&\forall x(\exists y(x = 2 \cdot y) \to (2 \cdot x \geq 0 \lor 3 \cdot x < 2)) \\
\leftrightarrow\, &\forall x(2 \mid x \to (2 \cdot x \geq 0 \lor 3 \cdot x < 2)) \\
\leftrightarrow\, &\neg\exists x \neg(2 \mid x \to (2 \cdot x \geq 0 \lor 3 \cdot x < 2)) \\
\leftrightarrow\, &\neg\exists x(2 \mid x \land 2 \cdot x < 0 \land 3 \cdot x \geq 2) \\
\leftrightarrow\, &\neg\exists x(2 \mid x \land 2 \cdot x \leq -1 \land 2 \leq 3 \cdot x) \\
\leftrightarrow\, &\neg\exists x(2 \mid x \land 6 \cdot x \leq -3 \land 4 \leq 6 \cdot x) \\
\leftrightarrow\, &\neg\exists z(12 \mid z \land z \leq -3 \land 4 \leq z) \leftrightarrow \bot
\end{aligned}$$

**Exercise 12.4.** **[Quantifier Elimination for $Th(\mathbb{N}, 0, S, =)$]**
Give a quantifier-elimination procedure for $Th(\mathbb{N}, 0, S, =)$ where $S$ is the successor operation on natural numbers, i.e. $S(n) = n + 1$.

*Hint*: $a = b$ iff $S^k(a) = S^k(b)$ for any $a, b, k \in \mathbb{N}$.

**Solution:**
We assume $F = \exists x(A_1 \wedge \ldots \wedge A_n)$ where $x$ occurs in all $A_i$ and each $A_i$ is of the form

$$S^k(x) = S^m(t) \text{ or } S^k(x) \neq S^m(t)$$

where $t$ is 0 or a variable (using symmetry of $=$).

If $x$ occurs on both sides of an atom $A_i$, we can compare the number of successors and replace it with $\perp$ or $\top$. Formally, $Th(\mathbb{N}, 0, S) \models (S^k(x) = S^k(x)) \leftrightarrow \top$ and $Th(\mathbb{N}, 0, S) \models (S^k(x) \neq S^k(x)) \leftrightarrow \perp$, and similarly for different numbers of successors.

Hence, we may further assume that $x \neq t$.

We have to distinguish two cases:

1. All $A_i$ only use $\neq$, but not $=$. Then return $\top$, because

$$Th(\mathbb{N}, 0, S) \models (\exists x(A_1 \wedge \ldots \wedge A_n)) \leftrightarrow \top$$

2. There is at least one $A_i$ of the form $S^m(x) = t$ where $x \neq t$.

   We replace $A_i$ as follows:

   - If $k > 0$, we add the constraints $t \neq 0 \wedge \ldots \wedge t \neq S^{m-1}(0)$ to ensure that the solution for $x$ is non-negative.
   - Otherwise, replace it with $\top$.

   The other $A_j$ $(i \neq j)$ can be replaced as follows: Let $A_j$ be $S^k(x) = u$. Using the hint, first increment both sides by $m$: $S^{k+m}(x) = S^m(u)$. Then, substitute $A_i$, resulting in $S^k(t) = S^m(u)$.

   This works similarly for inequality, resulting in $S^k(t) \neq S^m(u)$.

For optimization purposes, we could also assume that either side of the equalities/inequalities contains no successor application. If they do, we can decrement until at least one side is 0 or a variable.

**Homework 12.1.** **[Quantifier Elimination]** (6 points)
Perform Presburger arithmetic quantifier elimination (Cooper's algorithm) for each formula:

1. $\forall x \forall y (0 < y \land x < y \to x + 1 < 2 \cdot y)$
2. $\forall x (\exists y (x = 2 \cdot y \land 2 \mid y) \to 4 \mid x)$

**Homework 12.2.** **[Garnix Problem]** (6 points)
A university brewery has two kegging machines ($M_1$ and $M_2$) and produces two products, beer ($B$) and shandy ($R$). $M_1$ needs 50 minutes to fill a keg of beer but only 24 minutes to fill a keg of shandy, while $M_2$ needs 30 minutes for a keg of either one drink.

For a student festival beginning two weeks from now, an order of 75 kegs of beer and 95 kegs of shandy has been placed, which has to be fulfilled during the upcoming week. However, machine $M_1$ has to undergo maintance and therefore is expected to be running between 37 and 42 hours, while $M_2$ can be operated up to 100 hours during the whole week (and as few hours as desirable).

Use Fourier–Motzkin and Ferrante–Rackoff elimination to check wether the brewery can hold up to its promise.

**Homework 12.3.** **[Quantifier Elimination for $Th(\mathbb{N}, 0, S, =, <)$]** (8 points)
Similarly to Exercise 12.4, give a quantifier-elimination procedure for $Th(\mathbb{N}, 0, S, =, <)$.