

Basic Proof Theory

Propositional Logic

(See the book by Troelstra and Schwichtenberg)

Abbreviations

Until further notice:

\perp , \neg , \wedge , \vee , \rightarrow are primitives.

\top abbreviates $\neg\perp$

A possible simplification:

$\neg F$ abbreviates $F \rightarrow \perp$

Sequent Calculus rules

$$\frac{}{\perp, \Gamma \Rightarrow \Delta} \perp L$$

$$\frac{\Gamma \Rightarrow F, \Delta}{\neg F, \Gamma \Rightarrow \Delta} \neg L$$

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta} \wedge L$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta} \vee L$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad G, \Gamma \Rightarrow \Delta}{F \rightarrow G, \Gamma \Rightarrow \Delta} \rightarrow L$$

$$\frac{}{A, \Gamma \Rightarrow A, \Delta} Ax$$

$$\frac{F, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg F, \Delta} \neg R$$

$$\frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \wedge G, \Delta} \wedge R$$

$$\frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta} \vee R$$

$$\frac{F, \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \rightarrow G, \Delta} \rightarrow R$$

Notation: $\vdash_G S$ means that there is a proof tree for S using the above rules.

Admissible rules

Definition

A rule

$$\frac{S_1 \quad \dots \quad S_n}{S_{n+1}}$$

is **admissible** if $\vdash_G S_1, \dots, \vdash_G S_n$ together imply $\vdash_G S_{n+1}$.

Notation: $\vdash_n S$ means that there is a proof tree for S with depth $\leq n$.

Admissible rules

Lemma (Non-atomic Ax)

$\vdash_G F, \Gamma \Rightarrow F, \Delta$.

Lemma (Weakening)

If $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma', \Gamma \Rightarrow \Delta', \Delta$.

Admissible rules

Lemma (Inversion rules)

$\wedge L^{-1}$ If $\vdash_n F \wedge G, \Gamma \Rightarrow \Delta$ then $\vdash_n F, G, \Gamma \Rightarrow \Delta$

$\vee R^{-1}$ If $\vdash_n \Gamma \Rightarrow F \vee G, \Delta$ then $\vdash_n \Gamma \Rightarrow F, G, \Delta$

$\wedge R^{-1}$ If $\vdash_n \Gamma \Rightarrow F_1 \wedge F_2, \Delta$ then $\vdash_n \Gamma \Rightarrow F_i, \Delta$ ($i = 1, 2$)

$\vee L^{-1}$ If $\vdash_n F_1 \vee F_2, \Gamma \Rightarrow \Delta$ then $\vdash_n F_i, \Gamma \Rightarrow \Delta$ ($i = 1, 2$)

$\rightarrow R^{-1}$ If $\vdash_n \Gamma \Rightarrow F \rightarrow G, \Delta$ then $\vdash_n F, \Gamma \Rightarrow G, \Delta$

$\rightarrow L^{-1}$ If $\vdash_n F \rightarrow G, \Gamma \Rightarrow \Delta$
then $\vdash_n \Gamma \Rightarrow F, \Delta$ and $\vdash_n G, \Gamma \Rightarrow \Delta$

Admissible rules

Lemma (Contraction)

- ▶ *If $\vdash_n F, F, \Gamma \Rightarrow \Delta$ then $\vdash_n F, \Gamma \Rightarrow \Delta$*
- ▶ *If $\vdash_n \Gamma \Rightarrow F, F, \Delta$ then $\vdash_n \Gamma \Rightarrow F, \Delta$*

Lemma

If $\vdash_G \Gamma \Rightarrow \Delta$ then $\vdash_G \Gamma \Rightarrow \Delta - \{\perp\}$

Lemma (Atomic Cut)

If $\vdash_G \Gamma \Rightarrow A, \Delta$ and $\vdash_G A, \Gamma \Rightarrow \Delta$ then $\vdash_G \Gamma \Rightarrow \Delta$

Lemma (Cut)

If $\vdash_G \Gamma \Rightarrow F, \Delta$ and $\vdash_G F, \Gamma \Rightarrow \Delta$ then $\vdash_G \Gamma \Rightarrow \Delta$

Natural Deduction rules

$$\frac{F \quad G}{F \wedge G} \wedge I$$

$$\frac{F \wedge G}{F} \wedge E_1 \quad \frac{F \wedge G}{G} \wedge E_2$$

$$\frac{\begin{array}{c} [F] \\ \vdots \\ G \end{array}}{F \rightarrow G} \rightarrow I$$

$$\frac{F \rightarrow G \quad F}{G} \rightarrow E$$

$$\frac{F}{F \vee G} \vee I_1 \quad \frac{G}{F \vee G} \vee I_2$$

$$\frac{\begin{array}{c} [F] \quad [G] \\ \vdots \quad \vdots \\ F \vee G \quad H \quad H \end{array}}{H} \vee E$$

$$\frac{\begin{array}{c} [\neg F] \\ \vdots \\ \perp \end{array}}{F} \perp$$

Natural Deduction rules

Rules for \neg are special cases of rules for \rightarrow :

$$\frac{\begin{array}{c} [F] \\ \vdots \\ \perp \end{array}}{\neg F} \neg I \qquad \frac{\neg F \quad F}{\perp} \neg E$$

Hilbert System

The rule $\rightarrow E$ together with the following axioms:

$$F \rightarrow G \rightarrow F \quad (A_1)$$

$$(F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H \quad (A_2)$$

$$F \rightarrow G \rightarrow F \wedge G \quad (A_3)$$

$$F \wedge G \rightarrow F \quad (A_4)$$

$$F \wedge G \rightarrow G \quad (A_5)$$

$$F \rightarrow F \vee G \quad (A_6)$$

$$G \rightarrow F \vee G \quad (A_7)$$

$$F \vee G \rightarrow (F \rightarrow H) \rightarrow (G \rightarrow H) \rightarrow H \quad (A_8)$$

$$(\neg F \rightarrow \perp) \rightarrow F \quad (A_9)$$

Convention: \rightarrow associates to the right:

$F \rightarrow G \rightarrow H$ means $F \rightarrow (G \rightarrow H)$