Logics Exercise		
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SS 2018	Exercise Sheet 2	17.04.2018

Submission of homework: Before tutorial on 24.04.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 2.1. **[Resolution Completeness]**

1. Does $F \models C$ imply $F \vdash_{\text{Res}} C$? Proof or counterexample!
2. Can you prove $F \models C$ by resolution?

Exercise 2.2. **[Resolution of Horn-Clauses]**
Can the resolvent of two Horn-clauses be a non-Horn clause?

Exercise 2.3. **[Optimizing Resolution]**
We call a clause C *trivially true* if $A_i \in C$ and $\neg A_i \in C$ for some atom A_i. Show that the resolution algorithm remains complete if it does not consider trivially true clauses for resolution.

Exercise 2.4. **[Finite Axiomatization]**
Let M_0 and M be sets of formulas. M_0 is called *axiom schema* for M, iff for all assignments $\mathcal{A}$: $\mathcal{A} \models M_0$ iff $\mathcal{A} \models M$.

A set M is called *finitely axiomatized* iff there is a finite axiom schema for M.

1. Are all sets of formulas finitely axiomatized? Proof or counterexample!
2. Let $M = (F_i)_{i \in \mathbb{N}}$ be a sequence of formulas, such that for all i: $F_{i+1} \models F_i$, and not $F_i \models F_{i+1}$. Is M finitely axiomatized?

Exercise 2.5. **[Compactness Theorem]**
Suppose every finite subset of S is satisfiable. Show that then

every finite subset of $S \cup \{F\}$ is satisfiable or
every finite subset of $S \cup \{\neg F\}$ is satisfiable

for any formula F.

Homework 2.1. **[Resolution]** (4 points)
Use the resolution procedure to decide if the following formulas are satisfiable. Show your work (by giving the corresponding DAG or linear derivation)!

1. $(A_1 \vee A_2 \vee \neg A_3) \wedge \neg A_1 \wedge (A_1 \vee A_2 \vee A_3) \wedge (A_1 \vee \neg A_2)$
2. $(\neg A_1 \vee A_2) \wedge (\neg A_2 \vee A_3) \wedge (A_1 \vee \neg A_3) \wedge (A_1 \vee A_2 \vee A_3)$

Homework 2.2. **[Negative Resolution]** (6 points)
We call a clause C *negative* if it only contains negative clauses. Show that resolution remains complete if it only resolves two clauses if one of them is negative.

Homework 2.3. **[Satisfiability]** (5 points)
Check the following formulas for satisfiability using one of the algorithms seen in the lecture:

1. $(A \vee \neg B \vee \neg D \vee \neg E) \wedge (\neg B \vee C) \wedge B \wedge (\neg C \vee D) \wedge (\neg D \vee E)$
2. $\neg(((A \to B) \wedge (B \to A)) \to (A \leftrightarrow B))$
3. $(A \to E) \wedge (B \to \bot) \wedge (C \to B) \wedge (\top \to A) \wedge (A \wedge B \to C) \wedge (C \to D)$

Show your work! Remember to give a model for satisfiable formulas.

Homework 2.4. **[Application of the Compactness Theorem]** (5 points)
A finitely branching tree has the following structure:

- There is exactly one root node.
- Every node has a finite number of children.

We assign the root node the *level* 0 and the children of a node at level n the level $n+1$. Let T_n denote the set of all nodes at level n, and T the set of all nodes, i.e. $T = \bigcup_{n \in \mathbb{N}} T_n$. Let P_t for $t \in T$ be the set of parent nodes of a node, i.e. t is a child (or grand-child, ...) of all $t' \in P_t$. A path is a sequence of connected nodes, starting from the root node.

Prove the following lemma using the compactness theorem: Every countably infinite, finitely branching tree has an infinite path.

Hint: Use the following template for the proof.

1. Fix a set of tree nodes T. This set is (countably) infinite. You can assume that the sets T_n and the sets P_t are given.
2. For each node $t \in T$, let A_t be an atomic formula. If an assignment $\mathcal{A}$ makes A_t true, the node t is part of the path.
3. Define a set of propositions S that together guarantee the existence of an infinite path. That set is composed of three subsets:
   (a) For each level $n \in \mathbb{N}$, a node $t \in T_n$ is part of the path.
   (b) If a node t is part of the path, so are all of its parent nodes $t' \in P_t$.
   (c) For each level $n \in \mathbb{N}$, there is at most one node of level n part of the path.
4. Show that any finite subset of $S' \subseteq S$ is satisfiable by constructing an assignment such that $\mathcal{A}_{S'} \models S'$. Consider the largest n for which a proposition from subset (a) is contained in S'.
5. Hence, S is satisfiable. Show that a model $\mathcal{A} \models S$ represents an infinite path in T.