Logics Exercise

TU München
Institut für Informatik

Prof. Tobias Nipkow
Lars Hupel

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Submission of homework: Wednesday 02.05.2018, before noon; either via email or on paper in the TA's office (MI 00.09.063). Until further notice, homework has to be submitted in groups of two students.

Exercise 3.1. [**System G1c**]
An alternative definition of the sequent calculus ("G1c") is defined as follows:

Axioms

$$\text{Ax}\ \ A \Rightarrow A \qquad\qquad \text{L}\bot\ \ \bot \Rightarrow$$

Rules for weakening (W) and contraction (C)

$$\text{LW}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{RW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

$$\text{LC}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{RC}\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

Rules for the logical operators

$$\text{L}\wedge\ \frac{A_i, \Gamma \Rightarrow \Delta}{A_0 \wedge A_1, \Gamma \Rightarrow \Delta}\ (i = 0, 1) \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \text{R}\vee\ \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_0 \vee A_1}\ (i = 0, 1)$$

$$\text{L}\to\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \text{R}\to\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

Notably, weaking and contraction are built-in rules. Show that sequent calculus can be simulated by G1c, i.e., $\vdash_G \Gamma \Rightarrow \Delta$ implies $\vdash_{G1c} \Gamma \Rightarrow \Delta$.

Exercise 3.2. [**Cut Elimination, Semantically**]
Semantically prove the admissibility of the following rule:

$$\text{If } \vdash_G \Gamma \Rightarrow F, \Delta \text{ and } \vdash_G F, \Gamma \Rightarrow \Delta \text{ then } \vdash_G \Gamma \Rightarrow \Delta$$

Exercise 3.3. [**More Connectives**]
Define sequent rules for the logical connectives "nand" ($\bar{\wedge}$) and "xor" ($\otimes$).

Exercise 3.4. **[Intermediate Formulas]**
Let F, G be formulas such that $F \models G$. Prove that there is an *intermediate formula* H such that the following three conditions hold:

1. H contains only atomic formulas that occur in both F and G
2. $F \models H$
3. $H \models G$

How can H be constructed?

Homework 3.1. **[Sequent Calculus]** (2 points)
Prove the formula $((A \to \bot) \to A) \to A$ in System G1c.

Homework 3.2. **[Inversion Rules]** (6 points)
Show that the following inversion rules are admissible:

$$\frac{F \land G, \Gamma \Rightarrow \Delta}{F, G, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow F \to G, \Delta}{F, \Gamma \Rightarrow G, \Delta}$$

Homework 3.3. **[Sequent Prover]** (12 points)
Implement a sequent calculus prover in a high-level programming language, and test it for examples from this exercise sheet, the lecture, or your own.

Submission: Source code for prover and tests, `README` file containing instructions for how to build the prover and reproduce the tests; by email to `hupel@in.tum.de`. Allowed languages are: Haskell, OCaml, Java, Scala, Rust, Prolog, C++, Python. Only the standard library (i.e. no additional packages) may be used.