Logics Exercise		
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SS 2018	Exercise Sheet 5	08.05.2018

Submission of homework: Before tutorial on 15.05.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 5.1. **[Induction Proofs]**
Show that the following schema has a proof in natural deduction for all $n \geq 1$:

$$P_n = ((A_1 \wedge (A_2 \wedge (\ldots \wedge A_n) \ldots) \to B) \to (A_1 \to (A_2 \to (\ldots (A_n \to B) \ldots))))$$

Exercise 5.2. **[From Sequent Calculus to Natural Deduction]**
How can we construct, from a sequent calculus proof $\Gamma \Rightarrow \Delta$, a natural deduction proof $\Gamma \vdash_{\mathrm{N}} \bigvee \Delta$?

Exercise 5.3. **[Hilbert Calculus]**
Prove the following formula with a linear proof in Hilbert calculus: $(F \wedge G) \to (G \wedge F)$

Hint: Use the deduction theorem.

Homework 5.1. [A Smaller Set of Axioms for Hilbert Calculus] (5 points)
In the lecture, Hilbert calculus for propositional logic was introduced by means of nine axioms. However, the following three axioms are already sufficient:

A1 $F \to (G \to F)$

A2 $(F \to G \to H) \to (F \to G) \to F \to H$

A10 $(\neg F \to \neg G) \to (G \to F)$

Derive the following statement from the axioms above with help of $\to_E$: $\neg(F \to F) \to G$

Homework 5.2. [From Sequent Calculus to Natural Deduction] (8 points)
In Exercise 5.2, we constructed, from a sequent calculus proof of $\Gamma \Rightarrow \Delta$, a natural deduction proof $\Gamma \vdash_{\mathrm{N}} \bigvee \Delta$. That construction was "classical", because it required the use of the $\perp$ rule. We will consider a restricted Sequent Calculus here and the special case where $\Delta = \{F\}$, i.e., always contains exactly one formula.

The following system – a variant of Sequent Calculus – is called **G3c:**

$$\text{Ax} \quad P, \Gamma \Rightarrow P \ (P \text{ atomic}) \qquad\qquad \text{L}\perp \quad \perp, \Gamma \Rightarrow A$$

$$\text{L}\wedge \ \frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad\qquad \text{R}\wedge \ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\vee \ \frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \qquad\qquad \text{R}\vee \ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1} \ (i = 0, 1)$$

$$\text{L}\to \ \frac{A \to B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C} \qquad\qquad \text{R}\to \ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

Give a direct construction (without using the $\perp$ rule, but $\perp E$ is allowed) that transforms a G3c proof $\Gamma \Rightarrow F$ into a natural deduction proof $\Gamma \vdash_{\mathrm{N}} F$.

Hints: Perform an induction on the depth of the G3c proof. It is sufficient to consider the base cases and two other cases of your own choosing.

Homework 5.3. [Truth Tables] (7 points)
In the lecture, the following lemma was discussed:

Let $\text{atoms}(F) \subseteq \{A_1, \ldots, A_n\}$. Then we can construct a proof $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_{\mathrm{N}} F^{\mathcal{A}}$.

Recall the definition of $F^{\mathcal{A}}$:

$$F^{\mathcal{A}} = \begin{cases} F & \text{if } \mathcal{A}(F) = 1 \\ \neg F & \text{otherwise} \end{cases}$$

Prove the lemma by induction on F! It suffices to prove the cases for atomic formulas, negation, and disjunction.