Logics Exercise

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Submission of homework: Before tutorial on 03.07.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 12.1. **[Substitution in Sequent Calculus]**
Prove that $\vdash_G \Gamma \Rightarrow \Delta$ implies $\vdash_G \Gamma[t/x] \Rightarrow \Delta[t/x]$, where, for a set of formulas Γ, we define $\Gamma[t/x]$ to be $\{F[t/x] \mid F \in \Gamma\}$, i.e. free occurrences of x are replaced by t. Give two different proofs:

1. A syntactic proof, transforming the proof tree of $\vdash_G \Gamma \Rightarrow \Delta$.
2. A semantic proof, using correctness and completeness of $\vdash_G$.

Exercise 12.2. **[QE for DLO]**
Use the quantifier-elimination procedure for DLOs to check whether the following formula is a member of $Th(DLO)$:

$$\exists x \forall y \exists z((x < y \lor z < x) \land y < z)$$

Exercise 12.3. **[Fourier–Motzkin Elimination]**

1. $\exists x \exists y(2 \cdot x + 3 \cdot y = 7 \land x < y \land 0 < x)$
2. $\exists x \exists y(3 \cdot x + 3 \cdot y < 8 \land 8 < 3 \cdot x + 2 \cdot y)$

Exercise 12.4. **[Ferrante–Rackoff Elimination]**

$$\exists x(\exists y(x = 2 \cdot y) \to (2 \cdot x \geq 0 \lor 3 \cdot x < 2))$$

Homework 12.1. [Subtraction Logic] (8 points)
We consider a fragment of linear arithmetic, in which atomic formulas only take the form $x - y \leq c$ for variables x and y, and $c \in \mathbb{Q}$.

For a finite set S of such difference constraints, we can define a corresponding inequality graph $G(V, E)$, where V is the set of variables of S, and E consists of all the edges (x, y) with weight c for all constraints $x - y \leq c$ of S. Show that the conjuction of all constraints from S is satisfiable iff G does not contain a negative cycle.

How can you use this theorem to obtain a procedure for deciding whether a formula is a member of this fragment?

Homework 12.2. [Min, Max, Abs] (6 points)

1. Show that $\mathrm{Th}(\mathbb{R}, 0, 1, <, =, +, \min, \max)$ is decidable, where min and max return the minimum and maximum of two values.
2. Show that $\mathrm{Th}(\mathbb{R}, 0, 1, <, =, +, \min, \max, |\cdot|)$ is decidable, where $|\cdot|$ is the absolute value.

Homework 12.3. [Optimizing DLO] (6 points)
DLO suffers from a heavy performance loss because after each step, a DNF needs to be reconstructed. We want to study an optimization that may avoid this under some circumstances.

Assume that we want to eliminate an $\exists x F$ where

- F is closed (except for x),
- F contains no negations and quantifiers, and
- there are only lower or only upper bounds for x in F.

Then, $\exists x F \equiv \top$. Prove correctness of this optimization.