Logics Exercise		
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SS 2018	Exercise Sheet 6	15.05.2018

Submission of homework: Before tutorial on 23.05.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 6.1. [Equivalence]
Let F and G be arbitrary formulas. (In particular, they may contain free occurrences of x.) Which of the following equivalences hold? Proof or counterexample!

1. $\forall x(F \wedge G) \equiv \forall xF \wedge \forall xG$
2. $\exists x(F \wedge G) \equiv \exists xF \wedge \exists xG$

Solution:

1. Holds. Assume $\mathcal{A} \models \forall x(F \wedge G)$,
   $\iff$ for all $d \in U_{\mathcal{A}}$, we have $\mathcal{A}[d/x] \models F$ and $\mathcal{A}[d/x] \models G$,
   $\iff$ for all $d_1 \in U_{\mathcal{A}}$, we have $\mathcal{A}[d_1/x] \models F$ and for all $d_2 \in U_{\mathcal{A}}$, we have $\mathcal{A}[d_2/x] \models G$
   $\iff$ $\mathcal{A} \models \forall xF \wedge \forall xG$
2. Does not hold. Let $F = P(x)$ and $G = Q(x)$, $U_{\mathcal{A}} = \{0, 1\}$, $P^{\mathcal{A}} = \{0\}$, and $Q^{\mathcal{A}} = \{1\}$. Clearly, $\mathcal{A} \models \exists xF \wedge \exists xG$ but $\mathcal{A} \not\models \exists x(F \wedge G)$

Exercise 6.2. [Infinite Models]
Consider predicate logic with equality. We use infix notation for equality, and abbreviate $\neg(s = t)$ by $s \neq t$. Moreover, we call a structure finite iff its universe is finite.

1. Specify a finite model for the formula $\forall x\ (c \neq f(x) \wedge x \neq f(x))$.
2. Specify a model for the formula $\forall x \forall y\ (c \neq f(x) \wedge (f(x) = f(y) \longrightarrow x = y))$.
3. Show that the above formula has no finite model.

Solution:

1. $U^{\mathcal{A}} = \{0, 1, 2\} \subset \mathbb{N}$ and $c^{\mathcal{A}} = 0$ and $f^{\mathcal{A}}(0) = 1 \mid f^{\mathcal{A}}(n+1) = 2 - n$
2. $U^{\mathcal{A}} = \mathbb{N}$ and $c^{\mathcal{A}} = 0$ and $f^{\mathcal{A}}(n) = n + 1$
3. Assume a model $\mathcal{A}$. First note that the properties transfer to the semantic level, i.e., we have for all $x, y \in U_{\mathcal{A}}$:
$$c^{\mathcal{A}} \neq f^{\mathcal{A}}(x) \tag{1}$$
$$f^{\mathcal{A}}(x) = f^{\mathcal{A}}(y) \implies x = y \tag{2}$$
Now, we are in a position to show that $U_{\mathcal{A}}$ is infinite. For this, we define $x_i = (f^{\mathcal{A}})^i(c^{\mathcal{A}})$, i.e. i times $f^{\mathcal{A}}$ applied to $c^{\mathcal{A}}$. Clearly, we have $x_i \in U_{\mathcal{A}}$ for all i. We now show that $i < j$ implies $x_i \neq x_j$, immediately yielding infinity of $U_{\mathcal{A}}$. We do induction on i. For 0, we have $x_0 = c^{\mathcal{A}} \neq f^{\mathcal{A}}(\ldots) = x_j$ (by (1)). For $i + 1$, the induction hypothesis gives us $x_i \neq x_j$, which implies $x_{i+1} \neq x_{j+1}$ (by (2)). qed.

Exercise 6.3. [Skolem Form]
Convert the following formula into – in order – a rectified formula, closed and rectified formula, RPF and Skolem form.

$$P(x) \wedge \forall x\ (Q(x) \wedge \forall x \exists y\ P(f(x, y)))$$

Solution:

$$\begin{aligned}
& P(x) \wedge \forall x(Q(x) \wedge \forall x \exists y P(f(x, y))) & \\
\leadsto & P(x) \wedge \forall x_1(Q(x_1) \wedge \forall x_2 \exists y P(f(x_2, y))) & \text{rectified} \\
\leadsto & \exists x P(x) \wedge \forall x_1(Q(x_1) \wedge \forall x_2 \exists y P(f(x_2, y))) & \text{rectified and closed} \\
\leadsto & \exists x \forall x_1 \forall x_2 \exists y(P(x) \wedge (Q(x_1) \wedge P(f(x_2, y)))) & \text{RPF} \\
\leadsto & \forall x_1 \forall x_2(P(g) \wedge (Q(x_1) \wedge P(f(x_2, h(x_1, x_2))))) & \text{Skolem form}
\end{aligned}$$

Homework 6.1. [Predicate Logic] (6 points)

1. Specify a satisfiable formula F such that for all models $\mathcal{A}$ of F, we have $|U_{\mathcal{A}}| \geq 4$. You may or may not use equality.

2. Can you also specify a satisfiable formula F such that for all models $\mathcal{A}$ of F, we have $|U_{\mathcal{A}}| \leq 4$? Consider both predicate logic with and without equality.

Homework 6.2. [Skolem Form] (6 points)
Convert the following formulas into – in order – a rectified formula, closed and rectified formula, RPF and Skolem form.

1. $\forall x \exists y \forall z \exists w(\neg Q(f(x), y) \wedge P(a, w))$
2. $\forall z(\exists y(P(x, g(y), z)) \vee \neg \forall x\ Q(x))$

Homework 6.3. [Orders] (8 points)
A reflexive and transitive relation is called *preorder*. In predicate logic, preorders can be characterized by the formula

$$F \equiv \forall x \forall y \forall z\ (P(x,x) \wedge (P(x,y) \wedge P(y,z) \rightarrow P(x,z)))$$

1. Which of the following structures are models of F? Give an informal proof in the positive case and a counterexample for the negative case!
   (a) $U^{\mathcal{A}} = \mathbb{N}$ and $P^{\mathcal{A}} = \{(m,n) \mid m > n\}$
   (b) $U^{\mathcal{A}} = \mathbb{Z} \times \mathbb{Z}$ and $P^{\mathcal{A}} = \{((x,y),(a,b)) \mid\ a - x \leq b - y\ \}$
   (c) $U^{\mathcal{A}} = \mathbb{R}$ and $P^{\mathcal{A}} = \{(m,n) \mid m = n\}$

2. Let $Q(x,y)$ be specified as follows: $\forall x \forall y(P(x,y) \leftrightarrow Q(y,x))$. Assuming P is a preorder, is Q also a preorder?

3. Specify the notion of *equivalence relations*, that is, preorders that additionally satisfy symmetry.