Logics Exercise

TU München
Institut für Informatik

Prof. Tobias Nipkow
Lars Hupel

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Submission of homework: Before tutorial on 05.06.2018. Until further notice, homework has to be submitted in groups of two students.

Exercise 8.1. [Simultaneous substitution]
Recall that $F[t_1/x_1, \ldots, t_n/x_n]$ is the *simultaneous* substitution of $x_1, \ldots, x_n$ by $t_1, \ldots, t_n$.

1. Can we always express $F[t_1/x_1, \ldots, t_n/x_n]$ as a series of one-variable substitutions?
2. Can we always summarize a series of one-variable substitutions to a single simultaneous substitution?

Solution:

1. No. Counterexample: $F[y/x, x/y]$ (exchanges x and y).

   Note: It is possible for a concrete F though, because we could make up fresh variable names.

2. Yes. We can give a rule to "consolidate" a simultaneous substitution and a one-variable substitution:

$$F[t_1/x_1, \ldots, t_n/x_n][u/y] = F[t_1[u/y]/x_1, \ldots, t_n[u/y]/x_n, u/y]$$

   Apply this rule $n - 1$ times to obtain a single simultaneous substitution.

Exercise 8.2. [Occurs check]
What happens if one omits the occurs check in the unification algorithm? Find an example where a unification algorithm without occurs check diverges or returns the wrong result.

Solution:
Consider $x \stackrel{?}{=} f(x)$. Without the occurs check, we first produce $\sigma = \{x \mapsto f(x)\}$. The algorithm keeps going and produces $\sigma' = \{x \mapsto f(f(x))\}$, then $\sigma'' = \{x \mapsto f(f(f(x)))\}$ and so on.

Exercise 8.3. [Unifiable terms]
Specify the most general unifiers for the following sets of terms, if one exists:

$$L_1 = \{f(x, y), f(h(a), x)\}$$
$$L_2 = \{f(x, y), f(h(x), x)\}$$
$$L_3 = \{f(x, b), f(h(y), z)\}$$
$$L_4 = \{f(x, x), f(h(y), y)\}$$

Solution:

L_1 : $[h(a)/x, h(a)/y]$

L_2 : No unifier, occurs check fails on $x \sim h(x)$

L_3 : $[h(y)/x, b/z]$

L_4 : No unifier, occurs check fails on $h(y) \sim y$

Exercise 8.4. [Formulas without negation]
Prove that every predicate logic formula that only contains $\wedge, \vee, \forall, \exists, \longrightarrow$ and atomic formulas is satisfiable. Is such a formula also valid?

Solution:
Choose a suitable structure $\mathcal{A}$ with a singleton universe (e.g. $U_{\mathcal{A}} = \{0\}$) that interprets all predicates to be true everywhere. Additionally, assuming that the formula is rectified, introduce an interpretation for all free and bound variables to the only element of the universe. Then, by straightforward induction on the formula, we get that $\mathcal{A}$ is a model.

However, the formula needs not to be valid. Consider, e.g., the formula P for a nullary predicate P. This is clearly not valid, as there are models that interpret P not to hold.

Homework 8.1. **[Most general unifier]** (8 points)
Consider the unification problem $x \stackrel{?}{=} f(y)$. Without running the unification algorithm, prove that

1. $\sigma_1 = \{x \mapsto f(y)\}$ is a most general unifier.
2. $\sigma_2 = \{x \mapsto f(z), y \mapsto z\}$ is unifier, but not a most general unifier.

Hint: Argue using the definition of "most general unifier". Two substitutions σ and σ' can be proved equal by showing that they are equal on all variables, i.e., for all x, $x\sigma = x\sigma'$. Similarly, they can be proved not equal by demonstrating for a particular x that $x\sigma \neq x\sigma'$.

Homework 8.2. **[Unification]** (2 points)
Use the algorithm presented in the lecture to compute a most general unifier for the following set of formulas: $\{P(g(x), f(a)), P(y, x), P(g(f(z)), f(z))\}$

Homework 8.3. **[Untangling simultaneous substitution]** (5 points)
Recall Exercise 8.1. Demonstrate how to "untangle" a simultaneous substitution that has been obtained by consolidating one-variable substitutions back into one-variable substitutions.

Homework 8.4. **[Ground resolution]** (5 points)
Execute ground resolution to show that the following formula is unsatisfiable:

$$\forall x \forall y((P(x) \land \neg Q(y, y)) \to Q(x, y)) \land \neg \exists x(P(x) \land \exists y(Q(y, y) \land Q(x, y))) \land \exists y(P(y))$$