Foundations of Mathematics and Grundlagenkrise

Vincent Steffan

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Introduction

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- For this he used a collection of postulates and axioms.

Postulates

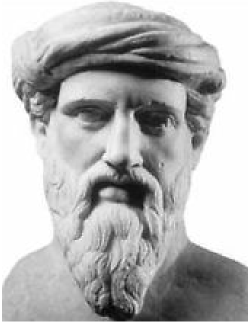
"Let the following be postulated:

- To draw a straight line from any point to any point.
- To extend a finite straight line continuously in a straight line.
- To describe a circle with any center and distance, the radius.
- That all right angles are equal to one another.
- (The so-called parallel postulate)That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

Axioms

- Things that are equal to the same thing are also equal to one another.
- If equals are added to equals, then the wholes are equal.
- If equals are subtracted from equals, then the remainders are equal.
- Things that coincide with one another are equal to one another.
- The whole is greater than the part.

A first example of a "Grundlagenkrise"



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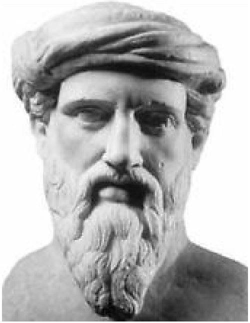

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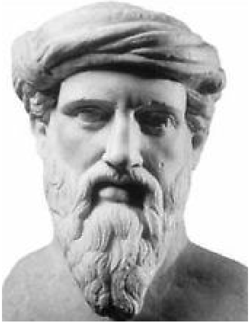

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- An example: If you take a square with side length 1, the diagonal has length $\sqrt{2}$, which is irrational.

Grundlagenkrise – The foundational crisis

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- He did this by establishing set theory in an axiomatic way.

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- Given any set F of nonempty pairwise disjoint sets, there is a set that contains exactly one member of each set in F.

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- Around the year 1900, Cesare Burali-Forti, Greogor Cantor himself and Bertrand Russell found three paradoxa.

Cantors paradox: Cardinal numbers

Let $\mathcal{A} = \{1, ..., n\}$. We say $\mathcal{A}$ has cardinal number n.
Two sets $\mathcal{A}$ and $\mathcal{B}$ have the same cardinality, if there is a bijective function $f : \mathcal{A} \to \mathcal{B}$.
We say that the cardinality of $\mathcal{A}$ is greater or equal than the cardinality of $\mathcal{B}$ if there is a surjective function $f : \mathcal{A} \to \mathcal{B}$.
The cardinality of $\mathcal{A}$ is greater if it is greater of equal, but not equal to the cardinality of $\mathcal{B}$.

Cantors paradox: Cardinal numbers

Theorem. Let $\mathcal{A}$ be a set and denote by $2^{\mathcal{A}}$ its powerset, i.e. the set of all subsets of $\mathcal{A}$. Then the cardinality of $2^{\mathcal{A}}$ is greater than the cardinality of $\mathcal{A}$.

Cantors paradox

Let $\mathcal{A}$ be the set of all sets. Then – since all subsets of $\mathcal{A}$ are sets – we have $2^{\mathcal{A}} \subset \mathcal{A}$, which implies, that the cardinality of $\mathcal{A}$ is greater or equal than the cardinality of $2^{\mathcal{A}}$, which contradicts the theorem.

Schools of recovery

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- In order to get to a different foundation of mathematics, three main schools were developed by the leading mathematicians of this time.

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- The intuitionists did not believe in the principle of "tertium non datur".

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- They avoided all known paradoxa, but were not able to prove consistency or completeness of this theory.
- In this process some powerful tools were developed: As an example, Peano was the first one to use symbols like "$\in$" or "$\Rightarrow$".

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- The formalists worked with so-called formal systems: A formal system consists of:
  - A finite set of symbols to construct formulas
  - A decision procedure to decide whether a formula is true or not.
  - A set of formulas assumed to be true, so-called axioms.

Gödels incompleteness theorem

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In the 1930's, Kurt Gödel showed, that the goal to find a complete and consistent foundation of mathematics cannot be reached.

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- A third property of a formal system is to be "sufficiently powerful", which essentially means, that it can describe basic mathematical concepts as the natural numbers.
- The property "recursive enumerability" prohibits things like infinitely long proofs. Essentially this property means that every proof can be verified in a mechanical way (e.g. by a computer).

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- Theorem. Any sufficiently powerful, recursively enumerable formal system is either inconsistent or incomplete.
- Theorem. Any sufficiently powerful consistent formal system cannot prove its own consistency.